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Quantum curves and $\mathcal{D}$-modules

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Abstract: In this article we continue our study of chiral fermions on a quantum curve. This system is embedded in string theory as an I-brane configuration, which consists of D4 and D6-branes intersecting along a holomorphic curve in a complex surface, together with a B-field. Mathematically, it is described by a holonomic $\mathcal{D}$-module. Here we focus on spectral curves, which play a prominent role in the theory of (quantum) integrable hierarchies. We show how to associate a quantum state to the I-brane system, and subsequently how to compute quantum invariants. As a first example, this yields an insightful formulation of (double scaled as well as general Hermitian) matrix models. Secondly, we formulate $c = 1$ string theory in this language. Finally, our formalism elegantly reconstructs the complete dual Nekrasov-Okounkov partition function from a quantum Seiberg-Witten curve.

Keywords: D-branes, Integrable Hierarchies, Topological Strings

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Contents

1 Introduction

The study of two-dimensional conformal field theories on Riemann surfaces has had many fruitful applications in physics and mathematics. In many ways the field theory of a chiral free fermion is the most important and instructive example. In that case one considers a Riemann surface or algebraic curve Σ together with a line bundle $\mathcal{L}$ that comes equipped with connection A. The free fermion partition function computes the determinant of the twisted Dirac operator $\overline{\partial}_A$ coupled to the line bundle $\mathcal{L}$. This determinant has many interesting properties, e.g. the dependence of this determinant on the connection A is captured by the Jacobi theta-function. It is known for a long time that these chiral determinants are closely related to integrable hierarchies of KP-type [1–3]. In the simplest case this relation arises as follows. One picks a point $P \in \Sigma$ on the curve together with a local trivialization e^t of the line bundle around P. The ratio with respect to a reference connection A_0

$$\tau(t) = \frac{\det \overline{\partial}_A}{\det \overline{\partial}_{A_0}}$$

then becomes a so-called tau-function of the KP-hierarchy. In the Hamiltonian formulation one associates a state $|\mathcal{W}\rangle$ in the fermionic Fock space $\mathcal{F}$ to the line bundle on $\Sigma - P$. In the semi-infinite wedge representation of the Fock space this state can be considered as the wedge product of a basis that spans the space of holomorphic sections

$$\mathcal{W} = H^0(\Sigma - P, \mathcal{L}).$$

Similarly, a coherent state $|t\rangle$ is associated to the local trivialisation around P. Combining these two ingredients the tau-function can be written as

$$\tau(t) = \langle t|\mathcal{W}\rangle. \tag{1.1}$$

With the advent of matrix models it became clear that string theory can also give rise to solutions of KP-type of the form (1.1). More recently this connection to integrable hierarchies has been reformulated and generalized through the methods of topological strings [4, 5]. These string theory solutions are similar, but not equivalent, to the familiar geometric solutions coming from CFT that are sketched above. In particular the relevant Fock space state $|\mathcal{W}\rangle$ does not have a purely geometric interpretation as generated by a space of sections over a curve. Yet, in the string theory setting an algebraic curve Σ does appear. (Here it should be stressed that this curve is not a string world-sheet, but should be considered as (part of) the target space geometry.) But in this case there is an extra parameter: the string coupling constant λ. Only in the genus zero or classical limit $\lambda \to 0$ a geometric curve arises. There have been many indications that λ should be interpreted as some form of non-commutative deformation of the underlying algebraic curve. In the simplest cases Σ appears as an affine rational curve given by a relation of the form

$$F(x, y) = 0,$$

in the complex two-plane $\mathbb{C}^2$, with a (local) parametrization

$$x = p(z), \qquad y = q(z),$$

with p, q polynomials. Of course, p and q commute: $[p, q] = 0$. However, the string-type solutions with $\lambda \neq 0$ are characterized by quantities P and Q that no longer commute but instead satisfy the canonical commutation relation

$$[P, Q] = \lambda.$$

In this case clearly P, Q cannot be polynomials, but are represented as differential operators, i.e. polynomials in z and ∂_z. As we will point out in this paper a suitable concept to frame these solutions is a $\mathcal{D}$-module. Instead of classical curve in the (x, y)-plane, we should think of a *quantum curve* as an analogue in the non-commutative plane $[x, y] = \lambda$. If we interpret

$$y = -\lambda \frac{\partial}{\partial x},$$

one can identify such a quantum curve as a holonomic $\mathcal{D}$-module $\mathcal{W}$ for the algebra $\mathcal{D}$ of differential operators in x. Now there is a straightforward way in which such a $\mathcal{D}$-module gives rise to a solution of the KP-hierarchy. By definition $\mathcal{W}$ carries an action of both x and ∂_x. However we are free to ignore the second action, which leaves us with the structure of an $\mathcal{O}$-module, $\mathcal{O}$ being the algebra of functions in x. By applying the infinite-wedge construction to the module $\mathcal{W}$ we obtain in the usual way a state $|\mathcal{W}\rangle$ in the fermion Fock space. Roughly speaking, $\mathcal{W}$ can be considered as the space of local sections that can be continued as sections of a (non-commutative) $\mathcal{D}$-module, instead of sections of a line bundle over a curve. This set-up can be generalized in many ways and in this fashion several constructions in topological string theory, matrix models and integrable hierarchies can be connected. It is the purpose of this paper to explain the connections between these familiar ingredients from the $\mathcal{D}$-module perspective.

This paper is structured as follows:

In section 2 we introduce our notion of a quantum curve and provide a construction of a tau-function associated to it. This tau-function arises, in an appropriate sense, from a quantization of the Krichever correspondence described in section 2.1. The physical system relevant for this quantization consists of an intersecting brane configuration with B-field in string theory. It is introduced in [5] and reviewed in section 2.2. The D4 and D6-branes wrap an affine complex curve Σ that is embedded in a complex symplectic plane. Endpoints of the strings stretched between D4 and D6-branes appear as fermionic modes on Σ, which are quantized by the B-field. This turns the chiral fermions into sections of a so-called $\mathcal{D}$-module. In section 2.2 we explain in which sense a $\mathcal{D}$-module quantizes the spectral curve and in section 2.3 we discuss how one can associate a fermionic state $|\mathcal{W}\rangle$, and thus a tau-function, to such a $\mathcal{D}$-module.

In sections 3, 4 and 5 we analyse three physical systems in which above quantization is realized: respectively matrix models, $c = 1$ string theory and $\mathcal{N} = 2$ supersymmetric gauge theories. We find that a quantization of the underlying classical curve yields a differential system that determines the corresponding partition functions. In other words, we see how our formalism in section 2 gives a unifying picture of these topics in terms of a underlying quantum curve.

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2 Quantum curves and invariants

The main object of interest in this paper is a chiral fermion field living on a holomorphic quantum curve. This set-up is embedded in string theory as a configuration of D4 and D6-branes that intersect along a classical curve Σ. Turning on a B-field on the D6-brane quantizes the curve Σ. As was shown in [5] this intersecting brane configuration is closely related to topological string theory, supersymmetric gauge theory and matrix models. More precisely, it relates to the topological B-model on non-compact Calabi-Yau backgrounds of the form

$$X_\Sigma: \;\; uv - F(z,w) = 0,$$

that is modeled on an affine curve Σ defined by the equation $F(x,y)=0$. The topological string partition function admits an expansion

$$Z_{\text{top}}(t,\lambda) = \exp\left(\sum \lambda^{2g-2}\mathcal{F}_g\right)$$

in the topological coupling constant λ, whose classical contribution $\mathcal{F}_0$ captures the complex periods

$$X_i = \int_{A_i}\Omega, \quad \partial_i\mathcal{F}_0 = \int_{B_i}\Omega$$

of X_Σ, while the semi-classical contribution $\mathcal{F}_1$ is known to compute a chiral determinant

$$\exp\mathcal{F}_1 = \det\,\overline{\partial}_\Sigma \tag{2.1}$$

on Σ. All higher order $\mathcal{F}_g$'s give quantum corrections to these results.

As we alluded to in the introduction, the chiral determinant (2.1) has an elegant interpretation in terms of certain geometric solutions of the KP hierarchy, which is known as the Krichever correspondence. In this context the chiral determinant is known as a tau-function. On the other hand, the total topological string partition function is also known to represent a tau-function of a KP hierarchy, though in this case it doesn't have a similar geometric interpretation. The aim of this section is to propose a quantum analog of the Krichever correspondence, starting from a quantum curve. We conjecture that this prescription computes the all-genus topological string partition function.

In this section we start by reviewing the Krichever correspondence. We continue by reviewing the intersecting brane system and explain what we mean by a quantum curve. In the last subsection we line out our prescription to obtain invariants from such a quantum curve.

2.1 Krichever correspondence

In this section we review the geometric Krichever correspondence that underlies the genus 1 free energy $\mathcal{F}_1$ of the topological string. In the simplest scenario we start with a Riemann surface Σ with a single puncture P. We study a chiral fermion field

$$\psi(z) = \sum_{r\in\mathbb{Z}+1/2}\psi_r z^{-r-1/2}, \qquad \{\psi_r^\dagger,\psi_s\} = \delta_{r+s,0}$$

on Σ which is coupled to a line bundle $\mathcal{L}$. The Hilbert space $\mathcal{H}$ of this fermion field is built by acting on the Dirac vacuum $|0\rangle$ with the fermionic modes ψ_r and $\psi_r^\dagger$. The boundary conditions near P, i.e. the choice of local coordinates and a choice of local frame of $\mathcal{L}$ near P, are encoded in a coherent state

$$\langle t| = \langle 0|e^{\sum t_n \alpha_n},$$

where $\alpha_n = \sum_r : \psi_r^\dagger \psi_{n-r} :$ are the bosonized modes. The partition function of the fermion field sweeps out a state $|\mathcal{W}\rangle$ in the Hilbert space $\mathcal{H}$, and for a given choice t of boundary conditions it reads

$$\tau(t) = \langle t|\mathcal{W}\rangle. \tag{2.2}$$

The Krichever correspondence tells us precisely how to find the state $|\mathcal{W}\rangle$. Choosing z^{-1} as a local coordinate around P, we define a subspace

$$\mathcal{W} \equiv H^0(\Sigma - P, \mathcal{L}) \subset \mathbb{C}[z] \oplus \mathbb{C}[[z^{-1}]]. \tag{2.3}$$

By picking a semi-infinite basis $w_n = z^n(1 + \mathcal{O}(z^{-1}))$ of this subspace, it can be quantized into a fermionic state

$$|\mathcal{W}\rangle = w_1 \wedge w_2 \wedge w_3 \wedge \ldots \in \mathcal{H}. \tag{2.4}$$

on which the fermionic modes act as

$$\psi_r = \frac{\partial}{\partial z^{-r+1/2}}, \quad \psi_r^\dagger = z^{r-1/2} \wedge .$$

Any state $|\mathcal{W}\rangle$ that we obtain in this way looks like

$$|\mathcal{W}\rangle = g|0\rangle, \quad \text{where } g = \exp\left(\sum c_{nm}\psi_n\psi_m^\dagger\right) \in Gl(\infty). \tag{2.5}$$

All states of the above form parametrize an infinite Grassmannian, which is well-known to give an elegant geometric formulation of the KP integrable hierarchy. (A more detailed review of these issues can be found in appendix A, which will be useful later on.)

Important for now is that although one can associate a tau function to any element $|\mathcal{W}\rangle$ in the Grassmannian, as in equation (2.2), only a dense subset of subspaces $\mathcal{W}$ in the infinite Grassmannian of measure 0 allows for a geometric Krichever interpretation. This subset can be characterized as follows. Basically, a subspace $\mathcal{W}$ has a geometric origin when there is an algebra $\mathcal{A}$ such that

$$\mathcal{A} \cdot \mathcal{W} \subset \mathcal{W}, \tag{2.6}$$

with $\mathcal{A}$ being non-trivial, i.e. $\mathcal{A} \neq \mathbb{C}$. In this situation the underlying curve Σ can be defined in terms of its spectrum $\mathcal{A} = H^0(\Sigma - P, \mathcal{L})$ and the tau-function (2.2) has an interpretation as a fermionic determinant det $\overline{\partial}_\Sigma$.

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2.2 B-field, $\mathcal{D}$-modules and quantum curve

Our motivation for quantizing the Krichever correspondence comes from [5], where it is argued that topological string theory is related by a chain of dualities to a system of intersecting D4 and D6 branes in type IIA with a background B-field. We now review some aspects of this relation.

2.2.1 I-brane configuration

In the intersecting brane set-up a central role is played by a holomorphically embedded curve $\Sigma \subset \mathcal{B}$, given by the equation

$$\Sigma : \; F(z,w) = 0,$$

where $\mathcal{B} = \mathbb{C} \times \mathbb{C}$ (or possibly with either $\mathbb{C}$ replaced by $\mathbb{C}^*$) is parametrized by complex coordinates (z,w). We consider this curve in the type IIA background

$$\text{(IIA)} \quad \mathbb{R}^3 \times \mathcal{B} \times \mathbb{R}^2 \times S^1, \tag{2.7}$$

and place a D4-brane wrapping $\mathbb{R}^3 \times \Sigma$ and a D6-brane wrapping $\mathcal{B} \times \mathbb{R}^2 \times S^1$. These branes intersect over Σ. Fermions on Σ are realized by massless modes of the 4-6 strings. The supersymmetry of the system ensures holomorphicity. The supersymmetries act trivially on the chiral fermions, which constitute a topological subsector of the complete system.

Non-commutativity in this set-up is introduced by turning on a constant B-field along $\mathcal{B}$, with holomorphic part

$$B = \frac{1}{\lambda} dz \wedge dw. \tag{2.8}$$

It is realized on the worldvolume of the D6-brane.

By a chain of dualities presented in [5] this I-brane configuration relates to the background

$$\text{(IIA)} \quad \mathbb{R}^3 \times \widetilde{X} \times S^1, \tag{2.9}$$

with a $D6$-brane wrapping $\widetilde{X} \times S^1$. This setup is appropriate for a computation of Donaldson-Thomas invariants $DT(n,d)$, physically interpreted as BPS bound states of $n \in H_0(\widetilde{X},\mathbb{Z}) \cong \mathbb{Z}$ D0-branes and $d \in H_2(\widetilde{X},\mathbb{Z})$ D2-branes to the D6-brane. The generating function of these invariants

$$Z_{\text{qu}}(t,\lambda) = \sum_{n,d} DT(n,d)\, e^{-n\lambda} e^{d\cdot t} \tag{2.10}$$

is closely related to the A-model topological string partition function on the toric manifold $\widetilde{X}$ with the complexified Kähler class $t \in H^2(\widetilde{X})$

$$Z_{\text{top}}(t,\lambda) = \exp\left(-\frac{t^3}{6\lambda^2} - \frac{1}{24} t \cdot c_2(\widetilde{X}) \right) Z_{\text{qu}}(t,\lambda). \tag{2.11}$$

Following the duality chain mentioned above, the (holomorphic) parameter λ which initially specified a value of the B-field (2.8) acquires an interpretation of the topological

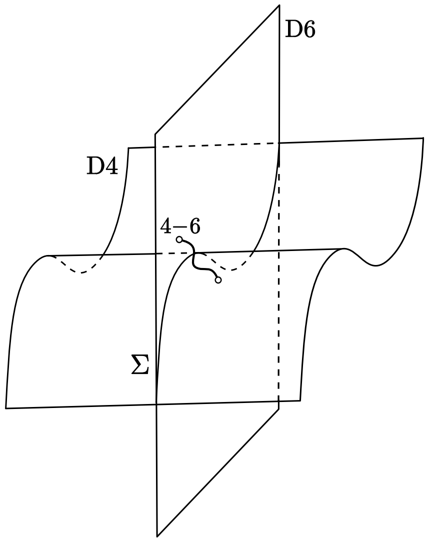


Figure 1. The I-brane configuration. A D4-brane intersects with a D6-brane along a curve Σ. The 4-6 string degrees of freedom show up as free fermions on Σ.

string coupling constant λ [5]. After summing over all bound states with p D4-branes as well, while weighting their contribution with a potential ξ, the partition function of the final configuration reads

$$Z_{\mathrm{I}}(\xi, t, \lambda) = \sum_{p \in H^2(\widetilde{X}, \mathbb{Z})} e^{p\xi} Z_{\mathrm{top}}(t + p\lambda, \lambda). \tag{2.12}$$

We identify this partition function with the I-brane partition function of the initial configuration (2.7).

The above system can also be easily related to the supersymmetric gauge theories leading to a system of D4-branes spanned between NS5-branes. As shown by Witten [6], such a configuration engineers $\mathcal{N} = 2$ supersymmetric gauge theories. We will elucidate this relation in much detail in section 5.

2.2.2 B-field

The B-field quantizes the fermions on Σ. Let us first repeat the general arguments of [5]. The algebra $\mathcal{A}$ of open 6-6 strings on the D6-brane describes the interaction (as illustrated in figure 2)

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \tag{2.13}$$

and is explicitly non-commutative in the presence of a B-field. The B-field introduces a gauge field A on the D6-brane that couples to the open strings and quantizes the algebra of zero-modes of those strings [7, 8]. With a B-field given by

$$B = \frac{1}{\lambda} dz \wedge dw \tag{2.14}$$

the non-commutativity parameter is λ. The complex coordinates z and w become non-commutative operators obeying

$$[z, w] = \lambda. \tag{2.15}$$

In case $\mathcal{B} = \mathbb{C}\times\mathbb{C}$ we can identify this algebra with the Weyl algebra of differential operators

$$\mathcal{A} \cong \mathcal{D}_{\mathbb{C}} = \langle z, \lambda\partial_z\rangle. \tag{2.16}$$

When we add to this system a D4-brane intersecting the D6-brane along the curve Σ, a 6-6 string acting on a 4-6 string can produce another 4-6 string (see figure 2)

$$\mathcal{A} \times \mathcal{M} \to \mathcal{M}. \tag{2.17}$$

This action endows the space of 4-6 open strings $\mathcal{M}$ with the structure of a module over the algebra $\mathcal{A}$ of 6-6 strings. Modules for the algebra of differential operators are called $\mathcal{D}$-modules.

To conclude, in the presence of a background flux, the chiral fermions on the I-brane should no longer be regarded as sections of the spin bundle $K^{1/2}$. Instead they should be viewed as sections of a $\mathcal{D}$-module.

In the context of string theory it is worth stressing the range of parameters α' and λ in which $\mathcal{D}$-module description is valid. The string coupling λ, which enters in the B-field flux as $B = \frac{1}{\lambda} dz \wedge dw$, plays an important role as quantization parameter. From the $\mathcal{D}$-module point of view there seems to be no restriction on λ, so one might hope that the $\mathcal{D}$-module even captures non-perturbative information. However, in a particular system under consideration some restrictions on the values of λ could arise that are related to the radius of convergence of the partition function. Although we do make some additional remarks in section 3 and in section 6, we do not study these issues in this paper.

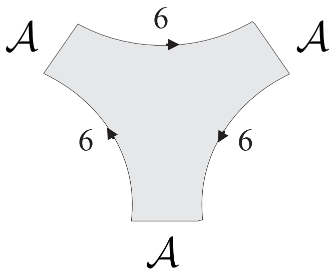
On the other hand, the string scale α' does not play a fundamental role in the $\mathcal{D}$-module. The $\mathcal{D}$-module describes the topological sector of the intersecting brane configuration, which is realized in terms of massless modes of the I-brane system. Therefore the $\mathcal{D}$-module description is valid only in the regime where α' is small (so that no massive modes interfere with our description). The most interesting case is of course when it is non-zero, as it provides a normalization factor for the worldsheet instanton contributions to the open 4-6 strings in the I-brane partition function (2.12). Section 5 clarifies this with an example.

2.2.3 $\mathcal{D}$-modules and quantum curves

We here introduce basic facts concerning $\mathcal{D}$-modules and explain why they naturally describe I-branes. More details concerning theory of $\mathcal{D}$-modules can be found in appendix B.

$\mathcal{D}$-modules are defined as modules for the algebra of differential operators $\mathcal{D}$. In this paper we are interested in $\mathcal{D}$-modules for the Weyl algebra $\mathcal{D} = \langle z, \partial_z\rangle$. These are affine $\mathcal{D}$-modules of rank 1 and can represented as

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D}\cdot P},$$



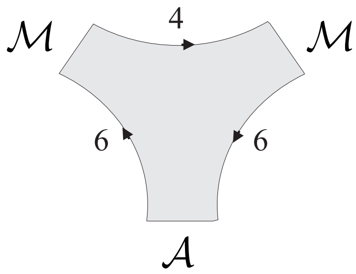


Figure 2. The algebra $\mathcal{A}$ of functions on Σ acts on the module $\mathcal{M}$ of free fermions. In the presence of a B-field the algebra $\mathcal{A}$ may be represented as a differential algebra, so that $\mathcal{M}$ becomes a $\mathcal{D}$-module.

where P is a linear differential operator $P = \sum_i a_i(z)\partial_z^i$. The module $\mathcal{M}$ therefore captures solutions to the differential equation

$$P\Psi = 0, \tag{2.18}$$

where Ψ takes values in some function space $\mathcal{V}$, for example the algebra $\mathcal{O}_{\mathbb{C}}$ of holomorphic functions on the complex plane $\mathbb{C}$. $\mathcal{D}$-modules of rank 1 are cyclic, i.e. they are generated by a single element $\Psi \in \mathcal{M}$, and so are of the form

$$\mathcal{M} = \{D\Psi : D \in \mathcal{D}\}. \tag{2.19}$$

To be more precise, $\mathcal{D}$-modules generated by the B-field (2.8) depend on λ and are known as $\mathcal{D}_\lambda$-modules [20] (when λ is considered as a formal variable). Since all the differential modules we consider are $\mathcal{D}_\lambda$-modules, we often omit the subscript λ.

The $\mathcal{D}$-module structure $\mathcal{D} \cdot \mathcal{M} \subset \mathcal{M}$ gives a quantization of the semi-classical description in equation (2.6). In particular, the rank 1 $\mathcal{D}$-module

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot P(z)} \tag{2.20}$$

is a quantization of the module

$$\mathcal{W} = \frac{\mathcal{O}_{\{z,w\}}}{\mathcal{O}_{\{z,w\}} \cdot F(z,w)} \tag{2.21}$$

of functions on the curve defined by $F(z,w) = 0$. We therefore refer to the underlying differential equation $P(z) = 0$ as a quantum curve. The I-brane set-up will obviously provide us with a rank 1 $\mathcal{D}$-module that represents a quantization of the I-brane curve Σ.[1]

Our notion of a quantum curve agrees with a notion of quantum spectral curves in the theory of (Hitchin) integrable systems. We discuss this relation shortly in appendix C. Here we just give some examples of $\mathcal{D}$-modules and their interpretation in terms of quantum curves.

[1] As a side remark notice that holonomic D-modules of dimension higher than 2 cannot be embedded in the 10 dimensions of string theory. Holonomic D-modules of dimension 2 are not related to the type II Calabi-Yau compactifications that we study in this paper, but could play a role in 4-dimensional Calabi-Yau compactifications in F-theory.

2.2.4 Examples

1) Take a linear partial differential operator on $\mathbb{C}$, for example

$$P = \lambda z \partial_z - 1. \tag{2.22}$$

The differential equation $P\Psi = 0$ is solved by $\Psi(z) = z^{1/\lambda}$. So according to (2.19) the corresponding $\mathcal{D}$-module can be represented as

$$\mathcal{M} = \langle z, \lambda\partial_z \rangle \, z^{1/\lambda}. \tag{2.23}$$

There are many equivalent ways of writing this module. For example, introducing $\widetilde{\Psi} = z\Psi$, the above differential equation is transformed into $\widetilde{P}\widetilde{\Psi} = 0$ with

$$\widetilde{P} = \lambda z \partial_z - \lambda - 1. \tag{2.24}$$

This new operator, as well as the solution to the new equation $\widetilde{\Psi} = z^{1+1/\lambda}$ look different than before. Nonetheless, they represent the same $\mathcal{D}$-module

$$\mathcal{M} = \langle z, \lambda\partial_z \rangle \, z^{1+1/\lambda} = \langle z, \lambda\partial_z \rangle \, z^{1/\lambda}. \tag{2.25}$$

This simple example illustrates how the formalism of $\mathcal{D}$-modules allows to study solutions to partial differential equations independently of the way in which they are written.

An equivalent way to study $\mathcal{D}$-modules is in terms of flat connections (see equation (B.10)). The flat connection corresponding to P is given by

$$\nabla_A = \lambda \partial_z dz - \frac{1}{z} dz, \tag{2.26}$$

and determines $\Psi(z)$ as a local flat section. It is a λ-deformation of the degree 1 spectral cover

$$\Sigma: \quad w = \frac{1}{z}, \tag{2.27}$$

with $z, w \in \mathbb{C}^*$, together with the (meromorphic) 1-form

$$A = \frac{1}{z} dz. \tag{2.28}$$

This example enters string theory as the deformed conifold geometry describing the $c = 1$ string. We will come back to it in section 4.

2) All the modules that we will study in this paper are over $\mathbb{C}$ or $\mathbb{C}^*$. It is important that they may be of any rank though. Let us therefore also give a rank two example on the complex plane $\mathbb{C}$. The second order differential equation

$$P\Psi = (\lambda^2 \partial_z^2 - z)\Psi \tag{2.29}$$

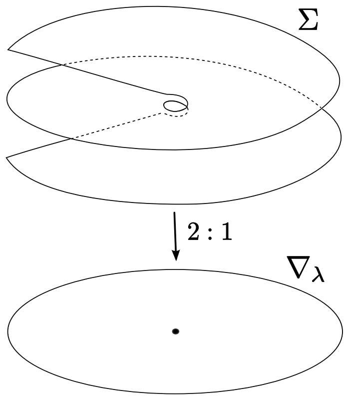


Figure 3. A second order differential operator P in $\lambda\partial_z$ defines a rank 2 λ-connection ∇_λ. The determinant of ∇_0 determines a degree 2 cover over $\mathbb{C}$ which is called the spectral curve Σ.

can be written equivalently as a rank two differential system

$$P_{ij}\psi_j = 0, \quad \text{with } P_{ij} = \begin{pmatrix} \lambda\partial_z & 0 \\ 0 & \lambda\partial_z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix}. \tag{2.30}$$

Holomorphic solutions of this linear system are captured by the map

$$\mathcal{M} = \frac{\mathcal{D}^{\oplus 2}}{\mathcal{D}^{\oplus 2}\, P_{ij}} \to \mathcal{O}_{\mathbb{C}}^{\oplus 2} \tag{2.31}$$

that sends the two generators $[(1,0)^t]$ and $[(0,1)^t]$ to two independent (2-vector) solutions of $P\Psi = 0$. The corresponding flat connection

$$\nabla_A = \partial_z dz - \frac{1}{\lambda}\begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz \tag{2.32}$$

is a λ-deformation of the degree 2 spectral cover (illustrated in figure 3)

$$\Sigma: \; w^2 = z, \tag{2.33}$$

with meromorphic 1-form $\eta = wdz|_\Sigma$. Note that this one-form pushes forward to the connection 1-form, or Higgs field,

$$A = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} dz \tag{2.34}$$

in the basis $\{dz, wdz\}$ of ramification 1-forms on the z-plane. Indeed, local sections of L push forward to local sections generated by 1 and w on the z-plane. Now, $wdz \cdot 1 = wdz$ and $wdz \cdot w = w^2 dz = zdz$ on Σ.

We will discuss the string theory interpretation of this $\mathcal{D}$-module in detail in section 3.

2.3 Prescription

Locally $\mathcal{D}$-module on a general curve is just a system of linear differential equations, that changes from patch to patch. It is therefore natural to try to associate quantum invariants locally, and glue them with the help of the $\mathcal{D}$-module transformations (which are a quantization of the usual coordinate transformations). This is exactly the strategy we follow in the examples treated in the following sections.

The simplest examples are deformations of affine curves of genus 0 with a single marked point at infinity, i.e. consisting of a single patch near infinity. In these examples we associate a quantum invariant to the curve in the following way:

We start with a differential equation $P(z)=0$ representing the quantum curve, where z^{-1} is the coordinate near infinity. Solutions to the differential equation $P(z)\Psi(z)=0$ form a module $\mathcal{M}$, which is in particular $\mathcal{O}$-module. We call this $\mathcal{O}$-module $\mathcal{W}$, and expect it to yield a subspace

$$\mathcal{W}\subset\mathbb{C}((z^{-1})).$$

In that case we can, analogously as in the semi-classical case, turn $\mathcal{W}$ into a fermionic state and compute a tau-function.

Since the I-brane configuration provides a $\mathcal{D}_\lambda$-module (in contrast to a $\mathcal{D}$-module) the resulting I-brane fermionic state $|\mathcal{W}\rangle$ is a λ-deformation as well. We conjecture that its determinant computes the all-genus topological string partition function, when the appropiate quantum curve is chosen.

In next sections we also discuss examples where we need to glue two local patches. Since each local patch yields a subspace of functions in the local variable, it is clear how the glueing should work: we need to insert a Fourier-like operator that relates the $\mathcal{D}$-module on the first patch to the one on the second patch. As a result, the partition function turns into a correlation function which contracts the two corresponding fermionic states, with the insertion of the corresponding Fourier-like operator. (This is similar as in [4].)

The above recipe doesn't tell us how to quantize a classical curve in a specific physical set-up. This is a very hard question in general. Moreover, it is not clear that the above prescription is independent of the chosen covering by local patches. Rather then providing a general theory, in the following sections we analyze several important examples of spectral curves in string theory, and determine $\mathcal{D}_\lambda$-modules that underlie their partition functions. Before we start with these examples though, let us explain the local procedure in more detail in two simple cases.

2.3.1 Examples

1) Let's first explain the rank 1 case, with a $\mathcal{D}_\lambda$-module on $\mathbb{C}$ specified by the (meromorphic) connection $\nabla_A=\partial_z-\frac{1}{\lambda}A(z)$ that may be trivialized as

$$\nabla_A=\partial_z-g_\lambda(z)^{-1}(\partial_z g_\lambda(z)).$$

When $g_\lambda(z)$ is a holomorphic function on $\mathbb{C}$ that equals $g_\lambda(0)=1$ at $z=0$ — in the notation of equation (A.20) this is an element of Γ_+ — this represents a pure gauge transformation on the disk, so that ∇_A corresponds to a regular flat connection on $\mathbb{C}$.

For any $g_\lambda(z)$ a fermionic section $\psi(z)$ of $\mathcal{L} \otimes K^{1/2}$ may be written as

$$\psi(z) = g_\lambda(z)\xi(z),$$

where $\xi(z)$ is a section of $\mathcal{L} \otimes K^{1/2}$ with trivial connection ∂_z. Flat sections $\Psi(z)$ are defined by the differential equation

$$\left(\partial_z - \frac{1}{\lambda}A(z)\right)\Psi(z) = 0.$$

They define a local trivialization of the bundle $\mathcal{L}$ with connection ∇_A, and we will use them to translate the geometric configuration into a quantum state.

A flat section for the trivial connection ∂_z is given by $\Xi(z) = 1$. We associate the pair $(\mathbb{C}, \partial_z)$ to the ground state $|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \ldots$. The gauge transformation $g_\lambda(z)$ maps the trivial solution $\Xi(z) = 1$ to $\Psi(z) = g_\lambda(z)$, which transforms the vacuum into the fermionic state

$$|\mathcal{W}\rangle = g_\lambda|0\rangle \equiv g_\lambda(z)z^0 \wedge g_\lambda(z)z^1 \wedge .$$

In other words, we build the quantum state by acting with the $\mathcal{D}$-module generator $\Psi(z) = g_\lambda(z)$ on the vacuum

$$\mathcal{W} = \mathcal{D}_\lambda \cdot \Psi(z).$$

The state $|\mathcal{W}\rangle$ is just the second quantization of the $\mathcal{D}_\lambda$-module $\mathcal{W}$. This state is non-trivial only when $g_\lambda(z)$ is not a pure gauge transformation (which would correspond to a Krichever solution). In this situation the flat section diverges near $z = 0$, corresponding to a distorted geometry in this region.

2) A degree n spectral curve Σ is quantized as a λ-connection of rank n. This is equivalent to a $\mathcal{D}_\lambda$-module $\mathcal{M}$ that is generated by a single degree n differential operator P. As an $\mathcal{O}_\mathbb{C}$-module, though, $\mathcal{M}$ is generated by an n-tuple

$$(\Psi(z), \partial_z\Psi(z), \ldots, \partial_z^{n-1}\Psi(z)),$$

where $\Psi(z)$ is a solution of the differential equation $P\Psi = 0$. In other words, this blends an n-vector of solutions to the linear differential system that the λ-connection defines. We will name this $\mathcal{O}_\mathbb{C}$-module

$$\mathcal{W} = \mathcal{O}_\mathbb{C} \cdot (\Psi(z), \partial_z\Psi(z), \ldots, \partial_z^{n-1}\Psi(z)) \ \subset \ \mathbb{C}((z^{-1}))$$

(of course it contains the same elements as $\mathcal{M}$). This is the subspace we want to second quantize into a fermionic state $|\mathcal{W}\rangle$.

P has n independent solutions Ψ_i that differ in their behaviour at infinity. These solutions have an asymptotic expansion around $z = \infty$ that contains a WKB-piece plus an asymptotic expansion in λ, and should thus be interpreted as perturbative solutions that live on the spectral cover. We suggest that the asymptotic expansion of any solution can be turned into a fermionic state that captures the all-genus I-brane partition function. This partition function thus depends on the choice of boundary conditions near z_∞.

Some of the WKB-factors will be exponentially suppressed near z_∞, while others grow exponentially. This depends on the specific region in this neighbourhood. The lines that characterize the changing behaviour of the solutions Ψ_i are called Stokes and anti-Stokes rays. Boundary conditions at infinity specify the solution up to a Stokes matrix: a solution that decays in that region can be added at no cost.

This implies that the perturbative fermionic state we assign to a $\mathcal{D}$-module depends on the choice of boundary conditions. On the other hand, the $\mathcal{D}$-module itself is independent of any of these choices and thus in some sense contains non-perturbative information and goes beyond the all-genus I-brane partition function. This agrees with the discussion in [27]. Nonetheless, the focus in this paper is on the perturbative information a $\mathcal{D}$-module provides.

3 Matrix model geometries

Hermitian one-matrix models with potential $W(x) = \sum_{j=0}^{d+1} u_j x^j$ are defined through the matrix integral

$$Z_N = \frac{1}{\mathrm{vol}(U(N))} \int DM \; e^{-\frac{1}{\lambda}\mathrm{Tr}\; W(M)}. \tag{3.1}$$

In the large N limit the distribution of the eigenvalues λ_i of M on the real axis becomes continuous and defines a hyperelliptic curve. This curve is called the spectral curve of the matrix model.

In the 't Hooft limit this matrix model has a dual description as the B-model topological string on Calabi-Yau geometries of the form [23]

$$uv + y^2 - W'(x)^2 + f(x) = 0, \tag{3.2}$$

where $f(x) = 4\mu \sum_{j=0}^{d-1} b_j x^j$ is a polynomial in x of degree $d-1$. The hyperelliptic curve Σ modeling the local threefold equals the matrix model spectral curve, with

$$f(x) = \frac{4\mu}{N} \sum_{i=1}^{N} \frac{W'(x) - W'(\lambda_i)}{x - \lambda_i}. \tag{3.3}$$

The potential $W(x)$ determines the positions of the cuts, containing the non-normalizable moduli, while the size of the cuts is determined by the polynomial $f(x)$, comprising the normalizable moduli $b_0, \ldots, b_{d-2}$ and the log-normalizable modulus b_{d-1}.

This duality may be generalized by starting with multi-matrix models, whose spectral curve is a generic (in contrast to hyperelliptic) curve in the variables x and y.

The I-brane picture suggests that the full B-model partition function on these Calabi-Yau geometries can be understood in terms of $\mathcal{D}$-modules. Even better, we will find that finite N matrix models are determined by an underlying $\mathcal{D}$-module structure.

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In the past, as well as recently, these matrix models have been studied in great detail in several contexts. Most importantly for us, it has been realized that a central role is played by the string or Douglas equation

$$[P, Q] = \lambda. \tag{3.4}$$

Here, P and Q are operators that implement multiplication with and differentiation with respect to a spectral coordinate. In a double scaling limit P and Q turn into differential operators. Physically, these critical models are known to describe minimal string theory.

Already in [24, 25] an attempt has been made to understand this string equation in terms of a quantum curve in terms of the expansion in the parameter λ. In Moore's approach this surface seemed to emerge from an interpretation of the string equation as isomonodromy equations.

In topological as well as minimal string theory a dominant role is played by holomorphic branes: either topological B-branes [4] or FZZT branes [26, 27]. Their moduli space equals the spectral curve, whereas the branes themselves may be interpreted as fermions on the quantized spectral curve. In particular, for $(p, 1)$ minimal models the so-called Lax operator P has been interpreted as the quantization of the spectral curve. In these string theories it is possible to compute correlation functions using a $W_{1+\infty}$-algebra [4, 28, 29] that implements complex symplectomorphisms of the complex plane $\mathcal{B}$ in quantum theory as Ward identities.

These advances strongly hint at a fundamental appearance of $\mathcal{D}$-modules in the theory of matrix models. Indeed, this section unifies recent developments in matrix models in the framework of section 2. Firstly, after a self-contained introduction in double scaled models we uncover the $\mathcal{D}$-module underlying the $(p, 1)$-models. In the second part of this section we shift our focus to general Hermitian multi-matrix models, and unravel their $\mathcal{D}$-module structure.

3.1 Double scaled matrix models and the KdV hierarchy

Our first goal is to find the $\mathcal{D}$-modules that explain the quantum structure of double scaled Hermitean matrix models. This double scaling limit is a large N limit in which one also fine-tunes the parameters to find the right critical behaviour of the multi-matrix model potential. Geometrically the double scaling limit zooms in on some branch points of the spectral curve that move close together. Spectral curves of double scaled matrix models are therefore of genus zero and parametrized as

$$\Sigma_{p,q}: \quad y^p + x^q + \ldots = 0. \tag{3.5}$$

The one-matrix model only generates hyperelliptic spectral curves, whereas the two-matrix model includes all possible combinations of p and q. These double scaled multi-matrix models are known to describe non-critical ($c < 1$) bosonic string theory based on the (p, q) minimal model coupled to two-dimensional gravity [30–34]. This field is therefore known as minimal string theory.

Zooming in on a single branch point yields the geometry

$$\Sigma_{p,1}: \quad y^p = x,$$

corresponding to the $(p, 1)$ topological minimal model. This model is strictly not a well-defined conformal field theory, but does make sense as 2d topological field theory. For $p = 2$ it is known as topological gravity [14, 35–37].

All (p, q) minimal models turn out to be governed by two differential operators

$$P = (\lambda\partial_x)^p + u_{p-2}(x)(\lambda\partial_x)^{p-2} + \ldots + u_0(x), \tag{3.6}$$

$$Q = (\lambda\partial_x)^q + v_{q-2}(x)(\lambda\partial_x)^{q-2} + \ldots + v_0(x), \tag{3.7}$$

of degree p and q respectively, which obey the string (or Douglas) equation

$$[P, Q] = \lambda. \tag{3.8}$$

P and Q depend on an infinite set of times $t = (t_1, t_2, t_3, \ldots)$, which are closed string couplings in minimal string theory, and evolve in these times as

$$\lambda\frac{\partial}{\partial t_j}P = [(P^{j/p})_+, P], \tag{3.9}$$

$$\lambda\frac{\partial}{\partial t_j}Q = [(P^{j/p})_+, Q], \tag{3.10}$$

The fractional powers of P define a basis of commuting Hamiltonians.[2] This integrable system is known as the p-th KdV hierarchy and the above evolution equations as the KdV flows.

The differential operator Q is completely determined as a function of fractional powers of the Lax operator P and the times t

$$Q = -\sum_{\substack{j\geq 1\\ j\neq 0 \mod p}} \left(1 + \frac{j}{p}\right) t_{j+p} P_+^{j/p}, \tag{3.13}$$

This implies that when we turn off all the KdV times except for $t_1 = x$ and fix t_{p+1} to be constant we find $Q = \lambda\partial_x$. This defines the $(p, 1)$-models

$$P = (\lambda\partial_x)^p - x, \quad Q = \lambda\partial_x. \tag{3.14}$$

One can reach any other (p, q) model by flowing in the times t.

The partition function of the p-th KdV hierarchy is a tau-function as in equation (1.1). The associated subspace $\mathcal{W} \in Gr$ may be found by studying the eigenfunctions $\psi(t, z)$ of the Lax operator P

$$P\psi(t, z) = z^p\psi(t, z). \tag{3.15}$$

[2]Notice that $L = P^{1/p}$ is a pseudo-differential operator, having an expansion

$$L = \lambda\partial_x + l_0(x) + l_1(x)(\lambda\partial_x)^{-1} + l_2(x)(\lambda\partial_x)^{-2} + \ldots, \tag{3.11}$$

in negative powers of $\lambda\partial_x$. This extended notion of a derivative is defined by the Leibnitz rule

$$\partial_x^n f = \sum_{k=0}^{\infty} \binom{n}{k} (\partial_x^k f)\partial^{n-k}, \tag{3.12}$$

for any $n \in \mathbb{Z}$ with $\binom{n}{k} = n \cdot \ldots \cdot (n - k + 1)/k!$. It gives the derivatives with $n < 0$ an interpretation of partial integration. L_+ is the notation for the restriction to the positive powers of L.

The so-called Baker function $\psi(t,z)$ represents the fermionic field that sweeps out the subspace $\mathcal{W}$ in the times t.

If we restrict to the $(p,1)$-models the Baker function $\psi(x,z)$ can be expanded in a Taylor series

$$\psi(x,z) = \sum_{k=0}^{\infty} v_k(z) \frac{x^k}{k!}. \tag{3.16}$$

Since $\psi(x,z)$ is an element of $\mathcal{W}$ for all times, this defines a basis $\{v_k(z)\}_{k\geq 0}$ of the subspace $\mathcal{W}$. In fact, it is not hard to see that the $(p,1)$ Baker function is given by the generalized Airy function

$$\psi(x,z) = e^{\frac{pz^{p+1}}{(p+1)\lambda}} \sqrt{z^{p-1}} \int dw \; e^{\frac{(-1)^{1/p+1}(x+z^p)w}{\lambda^{p/p+1}} + \frac{w^{p+1}}{p+1}}, \tag{3.17}$$

which is normalized such that its Taylor components $v_k(z)$ can be expanded as

$$v_k(z) = z^k(1 + \mathcal{O}(\lambda/z^{p+1})) \tag{3.18}$$

The $(p,1)$ model thus determines the fermionic state

$$|\mathcal{W}\rangle = v_0 \wedge v_1 \wedge v_2 \wedge \ldots, \tag{3.19}$$

where the $v_k(z)$ can be written explicitly in terms of Airy-like integrals (see [14] for a nice review). The invariance under

$$z^p \cdot \mathcal{W} \subset \mathcal{W} \tag{3.20}$$

characterizes this state as coming from a p-th KdV hierarchy. In the other direction, the state $|\mathcal{W}\rangle$ determines the Baker function (and thus the Lax operator) as the one-point function

$$\psi(t,z) = \langle t|\psi(z)|\mathcal{W}\rangle. \tag{3.21}$$

In the dispersionless limit $\lambda \to 0$ the derivative $\lambda\partial_x$ is replaced by a variable d, and the Dirac commutators by Poisson brackets in x and d. The leading order contribution to the string equation is given by the Poisson bracket

$$\{P_0, Q_0\} = 1, \tag{3.22}$$

where P_0 and Q_0 equal P and Q at $\lambda = 0$. The solution to this equation is

$$P_0(d;t) = x \tag{3.23}$$

$$Q_0(d;t) = y(x;t) \tag{3.24}$$

and recovers the genus zero spectral curve $\Sigma_{p,q}$ of the double scaled matrix model, parametrized by d. The KdV flows *deform* this surface in such a way that its singularities are preserved. (See the appendix of [27] for a detailed discussion.)

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Note that $\Sigma_{p,q}$ is not a spectral curve for the Krichever map. The Krichever curve is instead found as the space of simultaneous eigenvalues of the differential operators

$$[P, Q] = 0, \tag{3.25}$$

that is *preserved* by the KdV flow as a straight-line flow along its Jacobian. In fact, there is no such Krichever spectral curve corresponding to the doubled scaled matrix model solutions.

Wrapping an I-brane around $\Sigma_{p,q}$ quantizes the semi-classical fermions on the spectral curve $\Sigma_{p,q}$. The only point at infinity on $\Sigma_{p,q}$ is given by $x \to \infty$. The KdV tau-function should thus be the fermionic determinant of the quantum state $|\mathcal{W}\rangle$ that corresponds to this $\mathcal{D}$-module. In the next subsection we write down the $\mathcal{D}$-module describing the $(p, 1)$ model and show precisely how this reproduces the tau-function using the prescription outlined in section 2.

3.2 $\mathcal{D}$-module for topological gravity

We are ready to reconstruct the $\mathcal{D}$-module that yields the fermionic state $|\mathcal{W}\rangle$ in equation (3.19). For simplicity we study the $(2, 1)$-model, associated to an I-brane wrapping the curve

$$\Sigma_{(2,1)}: \qquad y^2 = x \qquad (\text{with } x, y \in \mathbb{C}). \tag{3.26}$$

Notice that this is an $2:1$ cover over the x-plane. It contains just one asymptotic region, where $x \to \infty$. Fermions on this cover will therefore sweep out a subspace $\mathcal{W}$ in the Hilbert space

$$\mathcal{W} \ \subset \ \mathcal{H}(S^1) = \mathbb{C}((y^{-1})), \tag{3.27}$$

the space of formal Laurent series in y^{-1}. The fermionic vacuum $|0\rangle \subset \mathcal{H}(S^1)$ corresponds to the subspace

$$|0\rangle = y^{1/2} \wedge y^{3/2} \wedge y^{5/2} \wedge \ldots, \tag{3.28}$$

which encodes the algebra of functions on the disk parametrized by y and with boundary at $y = \infty$. Exponentials in y^{-1} represent non-trivial behaviour near the origin and therefore act non-trivially on the vacuum state. In contrast, exponentials in y are holomorphic on the disk and thus act trivially on the vacuum.

The B-field $B = \frac{1}{\lambda} dx \wedge dy$ quantizes the algebra of functions on $\mathbb{C}^2$ into the differential algebra

$$\mathcal{D}_\lambda = \langle x, \lambda \partial_x \rangle. \tag{3.29}$$

Furthermore, it introduces a meromorphic connection 1-form $A = \frac{1}{\lambda} y dx$ on $\Sigma_{(2,1)}$, which pushes forward to the rank two λ-connection

$$\nabla_A = \lambda \partial_x - \begin{pmatrix} 0 & 1 \\ x & 0 \end{pmatrix} \tag{3.30}$$

on the base, parametrized by x. We claim that the corresponding $\mathcal{D}_\lambda$-module $\mathcal{M}$, generated by

$$P = (\lambda\partial_x)^2 - x, \tag{3.31}$$

describes the $(2,1)$ model. Let us verify this.

Trivializing the λ-connection ∇_A in (3.30) implies finding a rank two matrix $g(x)$ such that

$$\nabla_A = \lambda\partial_x - g'(x) \circ g^{-1}(x).$$

The columns of g define a basis of solutions $\Psi(x)$ to the differential equation $\nabla_A \Psi(x) = 0$. They are meromorphic flat sections for ∇_A that determine a trivialization of the bundle near $x = \infty$. As the connection ∇_A is pushed forward from the cover, $\Psi(x)$ is of the form

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi'(x) \end{pmatrix}.$$

Independent solutions have different asymptotics in the semi-classical regime where $x \to \infty$. In the $(2,1)$-model the two independent solutions $\psi_\pm(x)$ solve the differential equation

$$P\psi_\pm(x) = ((\lambda\partial_x)^2 - x)\psi_\pm(x) = 0. \tag{3.32}$$

Hence these are the functions $\psi_+(x) = \mathrm{Ai}(x)$ and $\psi_-(x) = \mathrm{Bi}(x)$, that correspond semi-classically to the two saddles

$$w_\pm = \pm\sqrt{x}/\lambda^{1/3}$$

of the Airy integral

$$\psi(x) = \frac{1}{2\pi i} \int dw\; e^{-\frac{xw}{\lambda^{2/3}} + \frac{w^3}{3}}. \tag{3.33}$$

The $\mathcal{D}$-module $\mathcal{M}$ can be quantized into a fermionic state for any choice of boundary conditions. Depending on this choice we find an $\mathcal{O}(x)$-module $\mathcal{W}_\pm$ spanned by linear combinations of $\psi_\pm(x)$ and of $\psi'_\pm(x)$. The fermionic state is generated by asymptotic expansions in the parameter λ of these elements.

The saddle-point approximation around the saddle $w_\pm = \pm\sqrt{x}/\lambda^{1/3}$ yields

$$\begin{aligned} \psi_\pm(x) &\sim y^{-1/2}\; e^{\mp\frac{2y^3}{3\lambda}} \left(1 + \sum_{n\geq 1} c_n \lambda^n (\pm y)^{-3n} \right) \\ &\sim y^{-1/2}\; e^{\mp\frac{2y^3}{3\lambda}}\; v_0(\pm y). \end{aligned}$$

To see the last step just recall the definition of $v_0(z)$ as being equal to the Baker function $\psi(x,z)$ evaluated at $x = 0$.[3] A similar expansion can be made for $\psi'(x)$ with the result

$$\psi'_\pm(x) \sim y^{1/2}\; e^{\mp\frac{2y^3}{3\lambda}}\; v_1(\pm y).$$

[3]Remark that x and z^2 appear equivalently in $\psi(x,z)$, while $\psi(x)$ and $\psi(x,z)$ only differ in the normalization term in z.

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Note that both expansions in λ are functions in the coordinate y on the cover. They contain a classical term (the exponential in $1/\lambda$), a 1-loop piece and a quantum expansion in λy^{-3}. When we restrict to the saddle $w = \sqrt{x}/\lambda^{1/3}$, these series blend the into the fermionic state

$$|\mathcal{W}_+\rangle = \psi_+(y) \wedge \psi'_+(y) \wedge y^p \psi_+(y) \wedge y^p \psi'_+(y) \wedge \dots. \tag{3.34}$$

Does this agree with the well-known result (3.19)?

First of all, notice that the basis vectors $x^k\psi(x)$ and $x^k\psi'(x)$, with $k > 0$, contain in their expansions the function $v_k(y)$ plus a sum of lower order terms in $v_l(y)$ (with $l < k$). The wedge product obviously eliminates all these lower order terms. Secondly, the extra factor $y^{-1/2}$ factors just reminds us that we have written down a fermionic state.

Furthermore, the WKB exponentials are exponentials in y and thus elements of the subgroup Γ_+ of holomorphic functions that extend over a disc centered around $y = 0$, whereas the expansions $v_k(y)/y^k$ are part of the subgroup Γ_- of functions that extend over a disc centered at $y = \infty$. (The definition of these subgroups and their action on the infinite Grassmannian can be found in appendix A.) Up to normal ordering ambiguities this shows that the WKB part gives a trivial contribution to the fermionic state $|\mathcal{W}_+\rangle$. In fact, the tau-function even cancels these ambiguities.

This shows that

$$|\mathcal{W}_+\rangle = v_0(y) \wedge v_1(y) \wedge v_2(y) \wedge \dots \tag{3.35}$$

is indeed the same as in (3.19), when we change variables from z to y in that equation. Of course, this doesn't change the tau-function.

So our conclusion is that the $\mathcal{D}$-module underlying topological gravity is the canonical $\mathcal{D}$-module

$$\mathcal{M} = \frac{\mathcal{D}_\lambda}{\mathcal{D}_\lambda((\lambda\partial_x)^2 - x)}. \tag{3.36}$$

This $\mathcal{D}$-module gives the definition of the quantum curve corresponding to the $(2,1)$ model and defines its quantum partition function in an expansion around $\lambda = 0$. Exactly the same reasoning holds for the $(p,1)$-model, where we find a canonical rank p connection on the base. It would be nice to be able to write down a $\mathcal{D}$-module for general (p,q)-models as well.

3.3 $\mathcal{D}$-module for Hermitean matrix models

$\mathcal{D}$-modules continue to play an important role in any Hermitean matrix model. In this subsection we are guided by [38] and [39, 40] of Bertola, Eynard and Harnad.

We first summarize how the partition function for a 1-matrix model defines a tau-function for the KP hierarchy. As we saw before, such a tau-function corresponds to a fermionic state $|\mathcal{W}\rangle$, whose basis elements we will write down. Following [38] we discover a rank two differential structure in this basis, whose determinant reduces to the spectral curve in the semi-classical limit. This $\mathcal{D}$-module structure is somewhat more complicated then the $\mathcal{D}$-module we just found describing double scaled matrix models.

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We continue with 2-matrix models, based on [40]. Instead of one differential equation, these models determine a group of four differential equations, that characterize the $\mathcal{D}$-module in the local coordinates z and w at infinity. The matrix model partition function may of course be computed in either frame.

3.3.1 1-matrix model

Let us start with the 1-matrix model partition function (3.1). By diagonalizing the matrix M the matrix integral may be reduced to an integral over the eigenvalues λ_i

$$Z_N = \int \prod_i d\lambda_i \; \Delta(\lambda)^2 \; e^{-\frac{1}{\lambda}\sum_i W(\lambda_i)}, \tag{3.37}$$

with the Vandermonde determinant $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j) = \det(\lambda_i^{j-1})$. The method of orthogonal polynomials solves this integral by introducing an infinite set of polynomials $p_k(x)$, defined by the properties

$$p_k(x) = x^k(1 + \mathcal{O}(x^{-1})), \tag{3.38}$$

$$\int dx \; p_k(x) \; p_l(x) \; e^{-\frac{1}{\lambda}W(x)} = h_k \delta_{k,l}. \tag{3.39}$$

The normalization of their leading term determines the coefficients $h_n \in \mathbb{C}$. Since the Vandermonde determinant $\Delta(x)$ is not sensitive to exchanging its entries x_i^{j-1} for $p_{j-1}(x_i)$, substituting $\Delta(x) = \det(p_{j-1}(x_i))$ turns the partition function into a product of coefficients

$$Z_N = N! \prod_{k=0}^{N-1} h_k. \tag{3.40}$$

With the help of orthogonal polynomials the large N behaviour of Z_N may be studied, while keeping track of $1/N$ corrections.

In this discussion the orthogonal polynomials are relevant since they build up a basis for the fermionic KP state. In an appendix of [38] it is shown that one should start at $t = 0$ with a state $|\mathcal{W}_0\rangle$ generated by the polynomials $p_k(x)$ for $k \geq N$

$$|\mathcal{W}_0\rangle = p_N(x) \wedge p_{N+1}(x) \wedge p_{N+2}(x) \wedge \ldots. \tag{3.41}$$

Notice that the vector $p_N(x)$ thus corresponds to the Fermi level and defines the Baker function in the double scaling limit. Acting on them with the commuting flow generated by

$$\Gamma_+ = \left\{ g(t) = e^{\sum_{n\geq 1} \frac{1}{n} t_n x^n} \right\} \tag{3.42}$$

defines a state $|\mathcal{W}_t\rangle = |g(t)\mathcal{W}_0\rangle$ at time t, which allows to compute a tau-function at time t. If the coefficients u_j in the potential $W(x)$ are taken to be $u_j = u_j^{(0)} + t_j$, this τ-function equals the ratio of the matrix model partition function Z_N at time t divided by that at $t = 0$.

Multiplying the orthogonal polyonomials by $\exp(-\frac{1}{2\lambda}W(x))$ doesn't change the fermionic state $\mathcal{W} = \mathcal{W}_0$ in a relevant way, since this factor is an element of Γ_+. To find the right $\mathcal{D}$-module structure, it is necessary to proceed with the quasi-polynomials

$$\psi_k(x) = \frac{1}{\sqrt{h_k}} p_k e^{-\frac{1}{2\lambda}W(x)}, \tag{3.43}$$

which form an orthonormal basis with respect to the bilinear form

$$(\psi_k, \psi_l) = \int dx\ \psi_k \psi_l. \tag{3.44}$$

It is possible to express both multiplication by x and differentiation with respect to x in terms of the basis of ψ_m's. The Weyl algebra $\langle x, \lambda\partial_x\rangle$ acts on these (quasi)-polynomials by two matrices Q and P

$$x\psi_k(x) = \sum_{l=0}^{\infty} Q_{kl}\psi_l \tag{3.45}$$

$$\lambda\partial_x\psi_k(x) = \sum_{l=0}^{\infty} P_{kl}\psi_l(x), \tag{3.46}$$

and the space of quasi-polynomials ψ_k is thus a $\mathcal{D}_\lambda$-module.

Notice that we anticipate that the $\mathcal{D}$-module possesses a rank two structure, since we started with a flat connection $A = \frac{1}{\lambda}y dx$ on an I-brane wrapped on a hyper-elliptic curve. Now, the matrices Q and P only contain non-zero entries in a finite band around the diagonal. The action of ∂_x on the semi-infinite set of $\psi_k(x)$'s can therefore indeed be summarized in a rank two differential system ([38] and references therein)

$$\lambda\partial_x \begin{bmatrix} \psi_N(x) \\ \psi_{N-1}(x) \end{bmatrix} = A_N(x) \begin{bmatrix} \psi_N(x) \\ \psi_{N-1}(x) \end{bmatrix}, \tag{3.47}$$

where $A_N(x)$ is a rather complicated 2×2-matrix involving the derivative W' of the potential and the infinite matrix Q:

$$A_N(x) = \frac{1}{2}W'(x)\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \gamma_N \begin{bmatrix} -\widetilde{W}'(Q,x)_{N,N-1} & \widetilde{W}'(Q,x)_{N,N} \\ -\widetilde{W}'(Q,x)_{N-1,N-1} & \widetilde{W}'(Q,x)_{N-1,N} \end{bmatrix}, \tag{3.48}$$

with

$$\widetilde{W}'(Q,x) = \left(\frac{W'(Q)-W'(x)}{Q-x}\right) \quad \text{and} \quad \gamma_N = \sqrt{\frac{h_N}{h_{N-1}}}. \tag{3.49}$$

Equation (3.47) is thus the rank two λ-connection defining the $\mathcal{D}_\lambda$-module structure on $\mathcal{W}$ that we were searching for! As a check, the determinant of this connection reduces to the spectral curve in the semiclassical, or dispersionless, limit [38]:

$$\Sigma_N: \quad 0 = \det\left(y 1_{2\times 2} - A_N(x)\right) \tag{3.50}$$

$$= y^2 - W'(x)^2 + 4\lambda \sum_{j=0}^{N-1}\left(\frac{W'(Q)-W'(x)}{Q-x}\right)_{jj} \tag{3.51}$$

(To make the coefficients in the above equation agree with (3.2), we rescaled $y \mapsto y/2$.) In conclusion we found the $\mathcal{D}$-module structure underlying Hermitean 1-matrix models.

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Remark that in the $N \to \infty$ limit we expect that the hyperelliptic curve defining the B-model Calabi-Yau (3.2) emerges from Σ_N. Indeed, in the 't Hooft limit Q corresponds classically to the coordinate x on the curve, whereas quantum-mechanically it is an operator whose spectrum is described by the eigenvalues λ_i of the infinite matrix M. In the large N limit we can therefore replace the matrix Q_{ij} in the definition for Σ_N by $\lambda_i \delta_{ij}$.

Note as well that we can rewrite the rank two connection for the vector $(\psi_N, \psi'_N)^t$ as

$$\lambda \partial_x \begin{bmatrix} \psi_N(x) \\ \psi'_N(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\det(A_N(x)) + \lambda Y & \lambda Z \end{bmatrix} \begin{bmatrix} \psi_N(x) \\ \psi'_N(x) \end{bmatrix}, \tag{3.52}$$

at least when $\mathrm{tr}(A_N(x)) = 0$, with Y and Z some derivatives of entries of $A_N(x)$. This brings the λ-connection in the familiar form of section 2. In the next subsection we clarify the differential structure in a simple example.

3.3.2 2-matrix model

Let us first say a few words on the $\mathcal{D}$-module structure underlying multi-matrix models, which capture spectral curves of any degree in x and y [39, 40]. The partition function for a two-matrix model, with two rank N matrices M_1 and M_2, is

$$Z_N = \int DM_1 DM_2 \; e^{-\frac{1}{\lambda}\mathrm{Tr}(W_1(M_1)+W_2(M_2)-M_1M_2)}, \tag{3.53}$$

where W_1 and W_2 are two potentials of degree d_1+1 and d_2+1. Choosing W_2 to be Gaussian reduces the 2-matrix model to a 1-matrix model. The 2-matrix model is solved by introducing two sets of orthogonal polynomials $\pi_k(x)$ and $\sigma_k(y)$. Again it is convenient to turn them into quasi-polynomials

$$\psi_k(x) = \pi_k(x) e^{-\frac{1}{\lambda}W_1(x)}, \quad \phi_k(y) = \sigma_k(y) e^{-\frac{1}{\lambda}W_2(y)}. \tag{3.54}$$

obeying the orthogonality relations

$$\int dxdy \; \psi_k(x)\phi_l(y) e^{\frac{xy}{\lambda}} = h_k \delta_{kl}. \tag{3.55}$$

Multiplying with or taking a derivative with respect to either x or y yields (just) two operators Q and P (and their transposes because of (3.55)), that form a representation of string equation $[P,Q] = 0$. Since Q is only non-zero in a band around the diagonal of size d_2+1 and P of size d_1+1, the quasi-polynomials may be folded into the vectors

$$\vec{\psi} = [\psi_N, \ldots, \psi_{N-d_2}]^t, \quad \vec{\phi} = [\phi_N, \ldots, \phi_{N-d_1}]^t. \tag{3.56}$$

Any other quasi-polynomial can be expressed as a sum of entrees of these vectors, with coefficients in the polynomials in x and y. These vectors are called windows. The differential operators $\lambda\partial_x$ and $\lambda\partial_y$ respect them, so that their action is summarized in a rank d_2+1 resp. rank d_1+1 λ-connection

$$\lambda \partial_x \vec{\psi}(x) = A_1(x)\vec{\psi}(x), \quad \lambda\partial_y \vec{\phi}(y) = A_2(y)\vec{\phi}(x). \tag{3.57}$$

This we interpret as two representations of the $\mathcal{D}_\lambda$-module underlying 2-matrix models. Indeed, [39] proves that the determinant of both differential systems equals the same spectral curve Σ, in the limit $\lambda \to 0$ when we replace $\lambda\partial_x \to y$ and $\lambda\partial_y \to x$. The defining equation of Σ is of degree d_1+1 in x and of degree d_2+1 in y.

In fact, it is useful to introduce two more semi-infinite sets of quasi-polynomials $\underline{\psi}_k(y)$ and $\underline{\phi}_k(x)$, as the Fourier transforms of $\psi_k(x)$ and $\phi_k(y)$ respectively. The action of the Weyl algebra on them may be encoded as the transpose of the above linear systems. The full system can therefore be summarized by (compare to (4.7))

$$\begin{aligned} x\text{-axis}: \quad & \{\psi_k(x),\ \underline{\phi}_k(x)\}, \quad \nabla_\lambda = \lambda\partial_x - A_1(x), \\ y\text{-axis}: \quad & \{\phi_k(y),\ \underline{\psi}_k(y)\}, \quad \nabla_\lambda = \lambda\partial_y - A_2(y). \end{aligned} \tag{3.58}$$

Moreover, the matrix model partition function can be rewritten as a fermionic correlator in either local coordinate

$$\begin{aligned} Z_N &= \int \prod_i d\lambda_i^1 d\lambda_i^2\ \Delta(\lambda^1)\Delta(\lambda^2)\ e^{-\frac{1}{\lambda}\sum_i (W_1(\lambda_i^1)+W_2(\lambda_i^2)-\lambda_i^1\lambda_i^2} \\ &= N! \prod_{k=0}^{N-1} \langle \psi_k(x)|\underline{\phi}_k(x)\rangle = N! \prod_{k=0}^{N-1} \langle \phi_k(y)|\underline{\psi}_k(y)\rangle \end{aligned} \tag{3.59}$$

with respect to the bilinear form in (3.44).

Furthermore, Bertola, Eynard and Harnad study the dependence on the parameters $u_j^{(1)}$ and $u_j^{(2)}$ appearing in the potentials W_1 and W_2. Deformations in these parameters leave the two sets of quasi-polynomials invariant as well. On $\vec{\psi}$ and $\vec{\phi}$ they act as matrices $U_j^{(1)}$ and $U_j^{(2)}$. This yields the 2-Toda system

$$\partial_{u_j^{(1)}} Q = -[Q, U_j^{(1)}] \qquad \partial_{u_j^{(1)}} P = -[P, U_j^{(1)}] \tag{3.60}$$

$$\partial_{u_j^{(2)}} Q = [Q, U_j^{(2)}] \qquad \partial_{u_j^{(2)}}, P = [P, U_j^{(1)}]. \tag{3.61}$$

In [39] it is proved that the linear differential systems (3.57) are compatible with these deformations, so that the parameters $u_j^{(1)}$ and $u_j^{(2)}$ in fact generate isomonodromic deformations. This shows precisely how the non-normalizable parameters in the potential respect the central role of the $\mathcal{D}_\lambda$-module (3.57) in the 2-matrix model.

3.4 Gaussian example

Let us consider the Gaussian matrix model with quadratic potential

$$W(x) = \frac{x^2}{2}, \tag{3.62}$$

that is associated to the spectral curve

$$y^2 = x^2 - 4\mu^2 \tag{3.63}$$

in the large N limit. In the Dijkgraaf-Vafa correspondence this matrix model is thus dual to the topological B-model on the deformed conifold geometry.

The Hermite functions

$$\psi_k^\lambda(x) = \frac{1}{\sqrt{h_k}} e^{-\frac{x^2}{4\lambda}} H_k^\lambda(x), \quad \text{with}$$
$$H_k^\lambda(x) = \lambda^{k/2} H_k\left(\frac{x}{\sqrt{\lambda}}\right) = x^k\left(1+\mathcal{O}\left(\frac{\sqrt{\lambda}}{x}\right)\right),$$

form an orthogonal basis for this model. Their inner product is given by

$$\int \frac{dx}{2\pi}\, \psi_k^\lambda(x)\psi_l^\lambda(x) = \lambda^k k! \sqrt{\frac{\lambda}{2\pi}}\delta_{kl} \quad \Rightarrow \quad h_k = \lambda^k k! \sqrt{\frac{\lambda}{2\pi}}.$$

The partition function of the Gaussian matrix model can be computed as a product of the normalization constants h_k. Using the asymptotic expansion of the Barnes function $G_2(z)$, that is defined by $G_2(z+1) = \Gamma(z)G_2(z)$, the free energy can be expanded in powers of λ

$$\begin{aligned}\mathcal{F}_N &= \log \prod_{k=1}^{N-1} h_k = \log\left(G_2(N+1)\frac{\lambda^{N^2/2}}{(2\pi)^{N/2}}\right) \\ &= \frac{1}{2}\left(\frac{\mu}{\lambda}\right)^2\left(\log\mu - \frac{3}{2}\right) - \frac{1}{12}\log\mu + \zeta'(-1) + \sum_{g=2}^{\infty}\frac{B_{2g}}{2g(2g-2)}\left(\frac{\lambda}{\mu}\right)^{2g-2},\end{aligned} \tag{3.64}$$

where B_{2g} are the Bernoulli numbers and $\mu = N\lambda$.

The derivatives of the Hermite functions are related as

$$\lambda\frac{d}{dx}\begin{bmatrix}\psi_k^\lambda(x)\\ \psi_{k-1}^\lambda(x)\end{bmatrix} = \begin{bmatrix}-x/2 & \sqrt{k\lambda}\\ -\sqrt{k\lambda} & x/2\end{bmatrix}\begin{bmatrix}\psi_k^\lambda(x)\\ \psi_{k-1}^\lambda(x)\end{bmatrix}. \tag{3.65}$$

So, according to the previous discussion, the $\mathcal{D}_\lambda$-module connection is given by

$$\lambda\frac{d}{dx} - A_N(x) = \lambda\frac{d}{dx} + \begin{bmatrix}x/2 & -\sqrt{N\lambda}\\ \sqrt{N\lambda} & -x/2\end{bmatrix}. \tag{3.66}$$

Here we choose $\vec{\psi} = [\psi_N, \psi_{N-1}]^t$ as window. In the large N limit the determinant of this rank two differential system indeed yields the spectral curve (3.63) with $\mu = \lambda N$.

Instead of using ψ_k^λ and ψ_{k-1}^λ as a basis, we can also write down the differential system for ψ_k^λ and its derivative $\psi'^\lambda_k(x) = \lambda\partial_x\psi_k^\lambda(x)$. Since this derivative is a linear combination of ψ_{k-1}^λ and $x\psi_k(x)$ (as we saw above), it is equivalent to use this basis to generate the fermionic state $\mathcal{W}$. We compute that

$$\lambda\frac{d}{dx}\begin{bmatrix}\psi_N^\lambda(x)\\ \psi'^\lambda_N(x)\end{bmatrix} = \begin{bmatrix}0 & 1\\ x^2-\lambda N-\lambda/2 & 0\end{bmatrix}\begin{bmatrix}\psi_N^\lambda(x)\\ \psi'^\lambda_N(x)\end{bmatrix}. \tag{3.67}$$

The spectral curve in the large N limit hasn't changed. Notice that in this form it is clear that the rank 2 connection is the push-forward of the connection $A = \frac{1}{\lambda}y dx$ on the spectral curve $y^2 = x^2 - 4\mu$ to the $\mathbb{C}$-plane, up to some λ-corrections.

In the double scaling limit the limits $N \to \infty$ and $\lambda \to 0$ are not independent as in the 't Hooft limit, but correlated, such that the higher genus contributions to the partition

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function are taken into account. In terms of the Gaussian spectral curve this limit implies that one zooms in onto one of the endpoints of the cuts. The orthogonal function $\psi_N^\lambda(x)$ turns into the Baker function $\psi(x)$ of the double scaled state $\mathcal{W}$.

In the Gaussian matrix model this is implemented by letting $x \to \sqrt{\mu} + \epsilon x$, where ϵ is a small parameter. So the double scaled spectral curve reads

$$y^2 = x, \tag{3.68}$$

while the differential system reduces to

$$\lambda \frac{d}{dx} \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \begin{bmatrix} \psi(x) \\ \psi'(x) \end{bmatrix}. \tag{3.69}$$

This is indeed the $\mathcal{D}$-module corresponding to the $(2,1)$-model.

4 Conifold and $c = 1$ string

The free energy (3.64) of the Gaussian matrix model pops up in the theory of bosonic $c = 1$ strings. This $c = 1$ string theory is formulated in terms of a single bosonic coordinate X, that is compactified on a circle of radius r in the Euclidean theory. A critical bosonic string theory (with $c = 26$) is obtained by coupling the above CFT to a Liouville field ϕ. The Liouville field corresponds to the non-decoupled conformal mode of the worldsheet metric. The local worldsheet action reads

$$\frac{1}{4\pi} \int d^2\sigma \left(\frac{1}{2} (\partial X)^2 + (\partial \phi)^2 + \mu e^{\sqrt{2}\phi} + \sqrt{2}\phi R \right),$$

where the coupling μ is seen as the worldsheet cosmological constant. In the Euclidean model there are only two sets of operators, that describe the winding and momenta modes of the field X. These vertex and vortex operators can be added to the action as marginal deformations with coefficients t_n and $\tilde{t}_n$.

Just like in $c < 1$ minimal string theories (the (p,q)-models of last section), the partition function of the $c = 1$ string is first computed using a dual matrix model description [41]. At the self-dual radius $r = 1$ it agrees with the Gaussian matrix model partition function in equation (3.64), where λ now plays the role of the $c = 1$ string coupling constant.

The matrix model dual to the $c = 1$ string is called matrix quantum mechanics. This duality is reviewed in much detail in e.g. [42–44]. Matrix quantum mechanics is described by a gauge field A and a scalar field M that are both $N \times N$ Hermitean matrices. The momentum modes of the $c = 1$ string correspond to excitations of M, whereas the winding modes are excitations of A. If we focus on the momentum modes, the (double scaled) matrix model is governed by the Hamiltonian

$$H = \frac{1}{2} \operatorname{Tr} \left(-\lambda^2 \frac{\partial^2}{\partial M^2} - M^2 \right).$$

Let us focus on solutions that depend purely on the eigenvalues λ_i of M. The Hamiltonian may be rewritten in terms of the eigenvalues as

$$H = \frac{1}{2}\Delta^{-1}(\lambda)\sum_i \left(-\lambda^2\frac{\partial^2}{\partial\lambda_i^2} - \lambda_i^2\right)\Delta(\lambda),$$

where $\Delta(\lambda)$ is Vandermonde determinant. It is convenient to absorb the factor Δ in the wavefunction solutions, making them anti-symmetric. Hence, the singlet sector of matrix quantum mechanics describes a system of N free fermions in an upside-down Gaussian potential.

To describe the partition function of the $c = 1$ model it is convenient to move over to light-cone coordinates $\lambda_\pm = \lambda \pm p$, so that elementary excitations of the $c = 1$ model are represented as collective excitations of free fermions near the Fermi level

$$\lambda_+\lambda_- = \mu. \tag{4.1}$$

When we restrict to $\lambda_\pm > 0$, scattering amplitudes can be computed by preparing asymptotic free fermionic states $\langle\tilde{t}|$ and $|t\rangle$ at the regions where one of $\lambda_\pm$ becomes very large.

In this picture the generating function of scattering amplitudes has a particularly simple form. It can be formulated as a fermionic correlator [45]

$$Z = \langle t|S|\tilde{t}\rangle, \tag{4.2}$$

where the fermionic scattering matrix $S \in GL(\infty, \mathbb{C})$ was first computed in [46]. Moreover, in [47] (see also Chapter V of [44]) and later in [4] it is noticed that S just equals the Fourier transformation

$$(S\psi)(\lambda_-) = \int d\lambda_+\, e^{\frac{1}{\lambda}\lambda_-\lambda_+}\psi(\lambda_+). \tag{4.3}$$

In the next section we show that this follows naturally from the perspective of $\mathcal{D}$-modules.

The result (4.2) shows that $c = 1$ string theory is an integrable system, just like the (p, q)-models in the last section. Since it depends on two sets of times this integrable system is not a KP system. Instead, the above expression defines a tau function of a 2-Toda hierarchy.

Notice that the Fermi level (4.1) is a real cycle on the complex curve

$$\Sigma: \quad zw = \mu, \tag{4.4}$$

which is a different parametrization of the spectral curve $y^2 = x^2 - \mu$ of the Gaussian 1-matrix model. In the revival of this subject a few years ago, a number of other matrix model interpretations have been found. This includes a duality with the Hermitean 2-matrix model, which makes the 2-Toda structure manifest [48], a Kontsevich-type model [49, 50] at the self-dual radius, and a so-called normal matrix model [51, 52], that parametrizes the dual real cycle on the complex curve Σ. Let us also mention that the well-known duality of the $c = 1$ string with the topological B-model on the deformed conifold [53], that follows, with a detour, from the more general Dijkgraaf-Vafa correspondence.

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4.1 $\mathcal{D}$-module description of the $c = 1$ string

This paragraph reproduces the $c = 1$ partition function (4.2) from a $\mathcal{D}$-module point of view. The discussion continues the line of thought in section 5.5 of [4] and in [5].

As we have just seen, the $c = 1$ string is geometrically characterized by the presence of a holomorphic curve in $\mathbb{C} \times \mathbb{C}$ defined by

$$\Sigma_{c=1} : \quad zw = \mu.$$

Let us consider an I-brane wrapping the curve $\Sigma_{c=1}$. When we assume z as local coordinate the curve quantizes into the differential operator

$$P = -\lambda z \partial_z - \mu. \tag{4.5}$$

It is amusing that the differential operator P appears as a canonical example in the theory of $\mathcal{D}$-modules (see e.g. [11]) in the same way as the $c = 1$ string is an elementary example of a string theory.

We recognize this example from section 2, where a $\mathcal{D}$-module was associated to the differential operator P. However, now it is important not to forget that there are *two* asymptotic points z_∞ and w_∞. Let us call their local neighbourhoods U_z and U_w, as local coordinates are z and w respectively. At both asymptotic points the I-brane fermions will sweep out an asymptotic state. The quantum partition function should therefore be constructed from two quantum states.

Before constructing these states for general λ, let us first consider the semi-classical limit $\lambda \to 0$. In this limit the I-brane degrees of freedom are just conventional chiral fermions on $\Sigma_{c=1}$. The genus 1 part $\mathcal{F}_1$ of the free energy is obtained as the partition function of these semi-classical fermions. It can be computed by assigning the Dirac vacuum

$$|0\rangle_z = z^{1/2} \wedge z^{3/2} \wedge z^{5/2} \wedge \dots$$

to U_z and likewise the conjugate state

$$|0\rangle_w = w^{1/2} \wedge w^{3/2} \wedge w^{5/2} \wedge \dots$$

to U_w. To compare these states, we need an operator S that relates z to $1/z$. The semi-classical partition can then be computed as a fermionic correlator ${}_w\langle 0|S|0\rangle_z$, with the result that

$$e^{\mathcal{F}_1} = {}_w\langle 0|S|0\rangle_z = \prod_{k \geq 0} \mu^{k+1/2}. \tag{4.6}$$

Using ζ-function regularization we find that this expression yields the familiar answer $\mathcal{F}_1 = -\frac{1}{12} \log \mu$.

In order to go beyond 1-loop, we should think in terms of $\mathcal{D}$-modules. Let us for a moment not represent their elements in terms of differential operators yet. In both asymptotic regions we then find the $\mathcal{D}$-modules

$$\begin{aligned} U_z : \quad & \mathcal{M} = \mathcal{D}/\mathcal{D}P, \quad \text{with} \quad P = \hat{z}\hat{w} - \mu, \\ U_w : \quad & \underline{\mathcal{M}} = \mathcal{D}/\mathcal{D}\underline{P}, \quad \text{with} \quad \underline{P} = \hat{w}\hat{z} - \mu + \lambda. \end{aligned}$$

Notice that the Weyl algebra $\mathcal{D} = \langle \hat{z}, \hat{w} \rangle$, with the relation $[\hat{z}, \hat{w}] = \lambda$, acts on monomials z^k and w^k in the module $\mathcal{M}$ as

$$
\begin{aligned}
\hat{z}(z^k) &= z^{k+1} \hat{z}(w^k) = \left(\lambda \partial_w + \frac{\mu - \lambda}{w}\right) w^k \\
\hat{w}(z^k) &= \left(-\lambda \partial_z + \frac{\mu}{z}\right) z^k \hat{w}(w^k) = w^{k+1}.
\end{aligned}
$$

Here, we just used the relation $\mathcal{D}P \equiv 0$ and wrote the elements in the basis $\{z^k, w^k \mid k \in \mathbb{Z}\}$ of $\mathcal{M}$. A basis of a representation of $\mathcal{M}$ on which $\hat{z}$ and $\hat{w}$ just act by multiplication by z resp. differentiation with respect to z is given by

$$
\begin{aligned}
v_k^z(z) &= z^k \cdot z^{-\mu/\lambda}, \\
v_k^w(z) &= \int dw \; e^{-zw/\lambda} \; w^{k-1} \cdot w^{\mu/\lambda}.
\end{aligned}
$$

Indeed, differentiation with respect to z clearly gives the same result as applying $\hat{w}$. Moreover, multiplying v_k^w by z gives

$$
z \cdot v_k^w(z) = \lambda \int dw \; e^{-zw/\lambda} \frac{\partial}{\partial w} \left(w^{k-1+\mu/\lambda} \right) = (\mu + \lambda(k-1)) v_{k-1}^w.
$$

Similarly, in the module $\underline{M}$ one can verify that

$$
\begin{aligned}
\hat{w}(w^k) &= w^{k+1} \hat{w}(z^k) = \left(-\lambda \partial_w + \frac{\mu}{w}\right) w^k \\
\hat{z}(w^m) &= \left(\lambda \partial_z + \frac{\mu - \lambda}{z}\right) z^k \hat{z}(z^k) = z^{k+1}.
\end{aligned}
$$

Hence in the representation of $\underline{\mathcal{M}}$ defined by

$$
\begin{aligned}
\underline{v}_k^w(w) &= w^{k-1} \cdot w^{\mu/\lambda}, \\
\underline{v}_k^z(w) &= \int dz \; e^{zw/\lambda} \; z^k \cdot z^{-\mu/\lambda},
\end{aligned}
$$

w and ∂_w act in the usual way.

Since we moved over to representations of the $\mathcal{D}$-module where the differential operator acts as we are used to, the S transformation, that connects the U_z and the U_w patch and thereby exchanges $\hat{z}$ and $\hat{w}$, must be a Fourier transformation. This is clear from the expressions for the basis elements w and $\tilde{w}$: S interchanges $v_k^z(z)$ with $\underline{v}_k^z(w)$, and $v_k^w(z)$ with $\underline{v}_k^w(w)$. In total we thus find the $\mathcal{D}$-module elements

$$
\begin{aligned}
U_z : &\quad v_k^z, \; v_k^w \\
U_w : &\quad \underline{v}_k^w, \; \underline{v}_k^z
\end{aligned}
\tag{4.7}
$$

Representing the $\mathcal{D}$-module in terms of differential operators of course gives the same result. A fundamental solution of $P\Psi(z) = 0$ is $\Psi(z) = z^{-\mu/\lambda}$, so that acting with $\mathcal{D} = \langle z, \partial_z \rangle$ on $\Psi(z)$ gives the elements v_k^z in $\mathcal{M}$. Likewise, we reconstruct the elements $\underline{v}_k^w$ from the fundamental solution of $\underline{P\Psi}(w) = 0$. Since $\mathcal{D} = \langle z, \partial_z \rangle$ and $\underline{\mathcal{D}} = \langle w, \partial_w \rangle$ are related by a Fourier transform, an element v_k of the $\mathcal{D}$-module in one asymptotic region is represented by its Fourier transform in the opposite region. This reproduces all elements in (4.7).

A λ-expansion of the $\mathcal{D}$-module element $\underline{v}^z_k$, using for example the stationary phase approximation, yields as zeroth order contribution

$$e^{\mu/\lambda}\left(\frac{\mu}{w}\right)^{k-\mu/\lambda},$$

while the subdominant contribution is given by

$$\sqrt{-\frac{2\pi\lambda\mu}{w^2}}.$$

So in total we find that

$$\underline{v}^z_k(w) = \sqrt{-2\pi\lambda}\;(\mu/e)^{-\mu/\lambda}\;w^{\mu/\lambda}\;\mu^{k+1/2}\;w^{-k-1}\;\psi_{\rm qu}\left(\frac{\mu}{w}\right).$$

This summarizes the contributions that we found before: the genus zero $w^{\mu/\lambda}$ and genus one $\mu^{k+1/2}w^{-k-1}$ results, plus the higher order contributions that are collected in $\psi_{\rm qu}$.

The all-genus partition function Z of this I-brane system can be easily computed exactly. Schematically it equals the correlation function

$$Z_{c=1} = \langle \mathcal{W}_w | S_\mu | \mathcal{W}_z \rangle,$$

where the S-matrix implements the Fourier transform between the two asymptotic patches. Similar to the arguments in (the appendices of) [47] and [4][4] we find that the result reproduces the perturbative expansion of the free energy as in equation (3.64). For completeness let us review the argument by comparing $\underline{v}^z_k(w)$ with $\underline{v}^w_k(w)$.

Notice that $\underline{v}^z_k(w)$ almost equals the gamma-function $\Gamma(z) = \int_0^\infty dt\; e^{-t}\; t^{z-1}$. Indeed, let us take the integration contour from $-i\infty$ to $i\infty$ and choose the cut of the logarithm to run from 0 to ∞. Then

$$\begin{aligned}
\underline{v}^z_k(w) &= \left(\frac{\lambda}{w}\right)\int_{-i\infty}^{i\infty} dz'\; e^{z'}\;\left(\frac{\lambda z'}{w}\right)^{k-\frac{\mu}{\lambda}} \\
&= \left(\frac{i\lambda}{w}\right)^{k+1-\frac{\mu}{\lambda}}\left[\int_{-\infty}^{0} dz'\; e^{iz'}\; e^{(k-\frac{\mu}{\lambda})\log z'} + \int_0^\infty dz'\; e^{iz'}\; e^{(k-\frac{\mu}{\lambda})\log z'}\right] \\
&= \left(\frac{i\lambda}{w}\right)^{k+1-\frac{\mu}{\lambda}}\left[\int_{i\infty}^{0} dz'\; e^{iz'}\; e^{(k-\frac{\mu}{\lambda})\log z'} + \int_0^{i\infty} dz'\; e^{iz'}\; e^{(k-\frac{\mu}{\lambda})\log z'}\right],
\end{aligned}$$

where we moved the contour along the positive imaginary axis. A change of variables and using that $\log(iz'-\epsilon) = \log z' - 3i\pi/2$ and $\log(iz'+\epsilon) = \log z' + i\pi/2$, for ϵ small and real, then yields

$$\underline{v}^z_k(w) = \left(\frac{i\lambda}{w}\right)^{k+1-\frac{\mu}{\lambda}}\left[e^{\pi i(k+1-\frac{\mu}{\lambda})/2} - e^{-3\pi i(k+1-\frac{\mu}{\lambda})/2}\right]\Gamma\left(k+1-\frac{\mu}{\lambda}\right).$$

[4]The argument presented in the appendix of [4] is not fully correct. The proper argument (as shown below) recovers a slightly different prefactor in front of the Gamma-function, related to the doubling in the appendix of [47].

which is the same as the theory of type II result in the appendix of [47]. Ignoring the exponential factor (which will only play a role non-perturbatively), we find that the free energy $\mathcal{F}$ equals the sum

$$\mathcal{F}\left(\lambda,\, \mu\right) = \sum_{k\geq 0} \left(k+1-\frac{\mu}{\lambda}\right)\log\lambda + \log\Gamma\left(k+1-\frac{\mu}{\lambda}\right).$$

It obeys the recursion relation

$$\mathcal{F}\left(\lambda,\, \mu+\frac{\lambda}{2}\right) - \mathcal{F}\left(\lambda,\, \mu-\frac{\lambda}{2}\right) = \left(\frac{1}{2}-\frac{\mu}{\lambda}\right)\log\lambda + \log\Gamma\left(\frac{1}{2}-\frac{\mu}{\lambda}\right).$$

which is known to be fulfilled by the $c = 1$ string (see for example appendix A in [55]), up to a term $-\frac{1}{2}\log(2\pi\lambda)$ that can be taken care of by normalizing the functions $\underline{v}_k$. The same result is found when analyzing the function v_k.

This concludes our discussion of the $c = 1$ string. It is the first $\mathcal{D}$-module example where we see how to handle curves with two punctures. The physical interpretation of the I-brane set-up furthermore provides a check of our formalism. Moreover, this example agrees with the claim that the $\mathcal{D}$-module partition function should be invariant under different parametrizations. Both the representation as $c = 1$ curve, $\Sigma_{c=1} : zw = \mu$, and that as a Gaussian matrix model spectral curve, $\Sigma_{mm} : y^2 = x^2+\mu$, yield the same partition function.

5 Seiberg-Witten geometries

Many times $\mathcal{N} = 2$ supersymmetric gauge theories proved to provide an important theoretical framework for testing new ideas in physics. It should be fair to say that the most important advances in this context are the solution of Seiberg and Witten in terms of a family of hyperelliptic curves, as well as the explicit solution of Nekrasov and Okounkov in terms of two-dimensional partitions. In what follows we will provide a novel perspective on these results, by wrapping an I-brane around a Seiberg-Witten curve. The B-field on the I-brane quantizes the curve, and a fermionic state is obtained from the corresponding $\mathcal{D}$-module. As we will see, this state sums over all possible fermion fluxes through the Seiberg-Witten geometry, and may be interpreted as a sum over geometries. First we briefly review the Seiberg-Witten and Nekrasov-Okounkov approaches.

The solution of the U(N) Seiberg-Witten theory is encoded in its partition function $Z(a_i, \lambda, \Lambda)$, which is a function of the scale Λ, the coupling λ and boundary conditions for the Higgs field denoted by a_i for $i = 1, \ldots, N$ (with $\sum_i a_i = 0$ for the SU(N) theory). The partition function is related to the free energy $\mathcal{F}$ as

$$Z(a_i, \lambda, \Lambda) = e^{\mathcal{F}} = e^{\sum_{g=0}^{\infty} \lambda^{2g-2}\mathcal{F}_g(a_i,\Lambda)}. \tag{5.1}$$

In the above expansion $\mathcal{F}_0$ is the prepotential which contains in particular an instanton expansion in powers of Λ^{2N}, while higher $\mathcal{F}_g$'s encode gravitational corrections. The U(N) Seiberg-Witten solution identifies the a_i's and the derivatives of the prepotential $\frac{1}{2\pi i}\frac{\partial \mathcal{F}_0}{\partial a_i}$ as the A_i and B_i periods of the meromorphic differential

$$\eta_{SW} = \frac{1}{2\pi i} v \frac{dt}{t} \tag{5.2}$$

on the hyperelliptic curve

$$\Sigma_{SW}: \quad \Lambda^N(t+t^{-1}) = P_N(v) = \prod_{i=1}^{N}(v-u_i). \tag{5.3}$$

Despite great conceptual advantages, extracting the instanton expansion of the prepotential from this description is a non-trivial task. However, an explicit formula for the partition function, encoding not only the full prepotential but also entire expansion in higher $\mathcal{F}_g$ terms, was postulated by Nekrasov in [54]. Subsequently this formula was derived rigorously jointly by him and Okounkov in [55] and independently by Nakajima and Yoshioka in [56, 57]. For U(N) theory this partition function is given by a sum over N partitions $\vec{R} = (R_{(1)}, \ldots, R_{(N)})$

$$Z(a_i, \lambda, \Lambda) = Z^{\text{pert}}(a_i, \lambda) \sum_{\vec{R}} \Lambda^{2N|\vec{R}|} \mu^2_{\vec{R}}(a_i, \lambda), \tag{5.4}$$

where

$$\mu^2_{\vec{R}}(a_i, \lambda) = \prod_{(i,m)\neq(j,n)} \frac{a_i - a_j + \lambda(R_{(i),m} - R_{(j),n} + n - m)}{a_i - a_j + \lambda(n-m)}, \tag{5.5}$$

$$Z^{\text{pert}}(a_i, \lambda) = \exp\Big(\sum_{i,j} \gamma_\lambda(a_i - a_j, \Lambda)\Big). \tag{5.6}$$

The function $\gamma_\lambda(x, \Lambda)$ is related to the free energy of the topological string theory on the conifold, and its various representations and properties are discussed extensively in [55] in appendix A. The vevs a_i are quantized in terms of λ, so that for $p_i \in \mathbb{Z}$,

$$a_i = \lambda(p_i + \rho_i), \qquad \rho_i = \frac{2i - N + 1}{2N}. \tag{5.7}$$

The approach of [54] is based on the localization technique in presence of the so-called Ω-background. In general this background provides a two-parameter generalization of the prepotential: the coupling λ is replaced by two geometric parameters ϵ_1 and ϵ_2. The prepotential, as given above, is recovered for $\lambda = \epsilon_1 = -\epsilon_2$. By the duality web of reference [5]supersymmetric gauge theories are related to intersecting brane configurations. The Nekrasov-Okounkov solution must therefore have an interpretation in terms of a quantum Seiberg-Witten curve, where λ plays the role of the non-commutativity parameter.

5.1 Dual partition functions and fermionic correlators

For a relation to the I-brane partition function (2.12), it is necessary to consider the dual of the partition function (5.4). This is introduced in [55] as the Legendre dual

$$Z^D(\xi, p, \lambda, \Lambda) = \sum_{\sum_i p_i = p} Z(\lambda(p_i + \rho_i), \lambda, \Lambda)\, e^{\frac{i}{\lambda}\sum_j p_j \xi_j}. \tag{5.8}$$

An important observation of Nekrasov and Okounkov is that this dual partition function can be elegantly written as a free fermion correlator. This is a consequence of the correspondence between fermionic states and two-dimensional partitions described in appendix A.

For U(1) there is no difference between the partition function and its dual and both can be written as

$$Z^D_{\mathrm{U}(1)}(p,\lambda,\Lambda) = \langle p|e^{-\frac{1}{\lambda}\alpha_1}\Lambda^{2L_0}e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle, \tag{5.9}$$

where $|p\rangle$ is the fermionic vacuum whose Fermi level is raised by $p = a/\lambda$ units and L_0 measures the energy of the state. A version of the boson-fermi correspondence implies the following decomposition

$$e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle = \sum_R \frac{\mu_R}{\lambda^{|R|}}|p;R\rangle \tag{5.10}$$

in terms of partitions R, where μ_R is the Plancherel measure

$$\mu_R = \prod_{1\le m<n<\infty} \frac{R_m - R_n + n - m}{n-m} = \prod_{\square\in R}\frac{1}{h(\square)} \tag{5.11}$$

which can be written equivalently as a product over hook lengths $h(\square)$.

For general N the dual partition function (5.8) looks very similar

$$Z^D_{\mathrm{U}(N)}(\xi_i;p,\lambda,\Lambda) = \langle p|e^{-\frac{1}{\lambda}\alpha_1}e^{H_{\xi_i}}\Lambda^{2L_0}e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle, \tag{5.12}$$

however, now this expression is obtained by blending N free fermions $\psi^{(i)}$ into a single fermion ψ, as explained in appendix A. In particular $H_{\xi_i} = \frac{1}{\lambda}\sum_r \xi_{(r+1/2) \bmod N}\psi_r\psi^\dagger_{-r}$, while the bosonic mode α_{-1} arises from the bosonization of the single blended fermion ψ. In formula (5.10) the Plancherel measure of a blended partition $\mathbf{R}$ can be decomposed into N constituent partitions as

$$\mu_{\mathbf{R}} = \sqrt{Z^{\text{pert}}(a_i,\lambda)}\,\mu_{\vec{R}}(a_i,\lambda), \tag{5.13}$$

with $\mu_{\vec{R}}$ and Z^{pert} given in (5.5) and (5.6). When read in terms of the N twisted fermions $\psi^{(i)}$, the correlator (5.12) involves a sum over the individual fermion charges p_i.

Our aim in this section is to derive the above fermionic expressions for the dual partition function from the perspective of this paper. In the next subsections we will see how first quantizing the Seiberg-Witten curve in terms of a $\mathcal{D}$-module elegantly reproduces to the fermionic correlators (5.9) and (5.12).

5.2 Fermionic correlators as $\mathcal{D}$-modules

In this section we compute the I-brane partition function for U(N) Seiberg-Witten geometries. We start with the simpler U(1) and U(2) examples and then generalize this to U(N). As a first principal step we notice that the U(N) Seiberg-Witten geometry

$$\Sigma_{SW}: \quad \Lambda^N(t+t^{-1}) = P_N(v) = \prod_{i=1}^N (v-u_i), \tag{5.14}$$

can be rewritten as

$$(P_N(v) - \Lambda^N t)(P_N(v) - \Lambda^N t^{-1}) = \Lambda^{2N}. \tag{5.15}$$

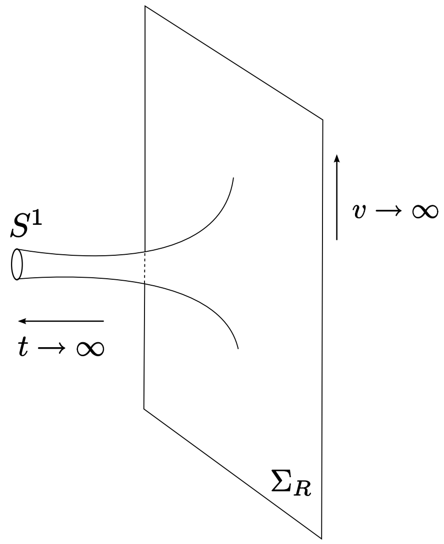

Figure 4. The right-half Seiberg-Witten geometry is distorted around the asymptotic point ($t \to 0, v \to \infty$). A fermion field on the quantized curve can be described as an element of a $\mathcal{D}$-module, and sweeps out a state $|\mathcal{W}\rangle$ at the S^1-boundary where $t \to \infty$.

This shows that the Seiberg-Witten surface may be seen as a transverse intersection of a left and a right half-geometry defined by

$$\Sigma_L: \ \Lambda^N t = P_N(v) \quad \text{resp.} \quad \Sigma_R: \ \Lambda^N t^{-1} = P_N(v), \tag{5.16}$$

which are connected by a tube of size Λ^{2N}. The left geometry parametrizes the asymptotic region where both $t \to \infty$ and $v \to \infty$, whereas the right geometry describes the region where $v \to \infty$ while $t \to 0$. This is illustrated in figure 4.

Next we wish to associate a subspace in the Grassmannian to both half Seiberg-Witten geometries. This will be swept out by a fermion field on the curve that couples to the holomorphic part of the B-field

$$B = \frac{1}{\lambda} ds \wedge dv \tag{5.17}$$

Since this B-field quantizes the coordinate v into the differential operator $\lambda\partial_s$, any subspace in this section is a $\mathcal{D}$-module for the differential algebra

$$D_{\mathbb{C}^*} = \langle t, \lambda\partial_s \rangle. \tag{5.18}$$

The free fermions on the Seiberg-Witten curves couple to the gauge field $A = \frac{1}{\lambda}\eta_{SW}$. This determines their flux through the A_i cycles of the Seiberg-Witten geometry as

$$p_i = \frac{1}{\lambda} \int_{A_i} \eta_{SW}. \tag{5.19}$$

The flux leaking through infinity is $p = \sum_{i=1}^N p_i$, which is zero for SU(N). A fermion field with fermion flux p at infinity, will sweep out a fermionic state in the pth Fock space. The

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parameters $\xi_i = \int_{B_i} \eta_{SW}$ are dual to the fermion fluxes. Notice that in the perturbative regime p_i can be written as a λ-expansion

$$\lambda p_i = u_i + \mathcal{O}(\lambda). \tag{5.20}$$

Since both half Seiberg-Witten geometries are distorted near $v = \infty$ (see figure 4), while a fermionic subspace can be read off in the neighbourhood where v is finite, both half-geometries parametrize a subspace of $\mathbb{C}((v))$:

$$\Sigma_L,\ \Sigma_R \subset \mathbb{C}((v)). \tag{5.21}$$

The trivial geometry corresponds to a disk with origin at $v = \infty$, whereas its boundary encloses the point $v = 0$. The vacuum state is therefore given by

$$|0\rangle = v^{-1/2} \wedge v^{-3/2} \wedge v^{-5/2} \wedge \dots. \tag{5.22}$$

Exponentials in v^{-1} act trivially (as pure gauge transformations in Γ_+) on this state, whereas exponentials in v transform the vacuum into a non-trivial fermionic state.

Finally, the partition function is recovered by contracting the left and the right fermionic state. Note that $s = -\log t$ is a local spatial coordinate on both half Seiberg-Witten geometries, which tends to $-\infty$ on the left and to $+\infty$ on the right. This makes a huge difference with the $c = 1$ geometry discussed in [4, 5], where the local coordinate is the exponentiated coordinate, which on the left is the inverse of that on the right. While in that example a non-trivial S-matrix is required to identify the left and right half-geometries, here we can just glue the fermionic states using the classic Hamiltonian L_0.

Let us now find these quantum states!

5.2.1 U(1) theory

The U(1) Seiberg-Witten curve is embedded in $\mathbb{C}^* \times \mathbb{C}$ as

$$\Lambda(t + t^{-1}) = v - u, \qquad (t = e^s \in \mathbb{C}^*,\ v \in \mathbb{C}) \tag{5.23}$$

where $u \in \mathbb{C}$ is a normalizable mode. This geometry may be factorized into a left and a right geometry

$$\Sigma_L :\ v = \Lambda t + u \quad \text{ and } \quad \Sigma_R :\ v = \Lambda t^{-1} + u, \tag{5.24}$$

that intersect transversely with degeneration parameter Λ^2.

The symplectic form $B = \frac{1}{\lambda} ds \wedge dv$ quantizes both half geometries into $\mathcal{D}_\lambda$-modules on a punctured disc $\mathbb{C}^*_t$, parametrized by t. We claim that these are characterized by the U(1) λ-connections

$$\nabla_L = -\lambda t \partial_t + \Lambda t + \lambda p \quad \text{ and } \quad \nabla_R = \lambda t \partial_t + \Lambda t^{-1} + \lambda p. \tag{5.25}$$

These are just the canonical quantizations of the classical Seiberg-Witten geometries, where additionally u is quantized into λp, with $p \in \mathbb{Z}$. They yield the linear differential equations

$$P_L \psi^\lambda_L(t;p) = \left(-\lambda t \partial_t + \Lambda t + \lambda p\right) \psi^\lambda_L(t;p) = 0, \tag{5.26}$$

$$P_R \psi^\lambda_R(t;p) = \left(\lambda t \partial_t + \Lambda t^{-1} + \lambda p\right) \psi^\lambda_R(t^{-1};p) = 0. \tag{5.27}$$

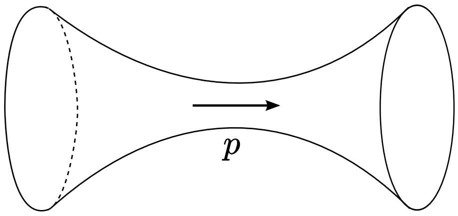

Figure 5. Contracting two Seiberg-Witten half-geometries yields the Nekrasov-Okounkov partition function corresponding to a fermion flux p through the surface.

The $\mathcal{D}_\lambda$-modules are of the canonical form

$$\mathcal{M}_{L/R} = \frac{\mathcal{D}_\lambda}{\mathcal{D}_\lambda \cdot P_{L/R}}, \tag{5.28}$$

and are generated by the solutions

$$\psi_L^\lambda(t;p) = t^p e^{\frac{\Lambda}{\lambda}t} \quad \text{and} \quad \psi_R^\lambda(t;p) = t^{-p} e^{\frac{\Lambda}{\lambda}t^{-1}}. \tag{5.29}$$

From the discussion in appendix A it follows that the factor t^{-p} acts on the right Dirac vacuum by raising the Fermi level into $|p\rangle$, while the exponent of t^{-1} translates to the exponentiated α_{-1} operator. With an analogous statement for the left state, the modules $\mathcal{M}_{L/R}$ translate into the Bogoliubov states

$$\langle \mathcal{W}_L| = \langle p| e^{\frac{\Lambda}{\lambda}\alpha_1} \qquad \text{and} \qquad |\mathcal{W}_R\rangle = e^{\frac{\Lambda}{\lambda}\alpha_{-1}}|p\rangle. \tag{5.30}$$

The U(1) Nekrasov-Okounkov partition function with fermion flux p (see figure 5) is found by contracting the above fermion states

$$Z_{NO}^\lambda(p;\Lambda) = \langle p| e^{\frac{\Lambda}{\lambda}\alpha_1} e^{\frac{\Lambda}{\lambda}\alpha_{-1}} |p\rangle. \tag{5.31}$$

The factors Λ can be pulled out of the exponentials by using the commutator $[L_0, \alpha_{\pm 1}] = \alpha_{\pm 1}$. Up to an extra factor $\Lambda^{-p^2/2}$ we find that

$$Z_{NO}^\lambda(p;\Lambda) \sim \langle p| e^{\frac{\alpha_1}{\lambda}} \Lambda^{2L_0} e^{\frac{\alpha_{-1}}{\lambda}} |p\rangle. \tag{5.32}$$

This has a nice geometrical explanation, since the left and right half geometries are connected by a tube of size Λ^2 as in the factorized form of the complete U(1) geometry. The factor Λ^{2L_0} is the Hamiltonian that describes the propagation of the fermion field along the tube. There is no need to generalize this standard-CFT factor, since both patches are described by the same space-coordinate s.

We also note that, as consistent with [4], the solution $\psi_R^\lambda(t;u)$ to $P_R\psi = 0$ equals the one-point-function

$$\langle p-1|\psi(t)|\mathcal{W}_R\rangle = \sum_n t^{-p-n} \langle p; R_n|\mathcal{W}_R\rangle = t^{-p} e^{\frac{\lambda}{\Lambda}t^{-1}} = \psi_R^\lambda(t;u), \tag{5.33}$$

where R_n represents a Young tableau consisting of just one row of n boxes.

5.2.2 U(2) theory

We apply now the above strategy for the U(2) geometry. We split the corresponding curve into a left and a right half geometry, and for brevity focus just on the right part defined by

$$\Sigma_R : \quad \Lambda^2 t^{-1} = (v - u_2)(v - u_1). \tag{5.34}$$

The B-field quantizes this equation into the second order differential equation

$$P_R \psi(t) = \left\{ \lambda^2 (t\partial_t - p_2)(t\partial_t - p_1) - \Lambda^2 t^{-1} \right\} \psi(s) = 0. \tag{5.35}$$

A change of variables $z = 2t^{-1/2}$ followed by the ansatz $\psi(z) = z^{-(p_1+p_2)}\phi(z)$ and the rescaling $z \mapsto (\lambda/\Lambda)z$ transforms this differential equation into the familiar Bessel equation

$$\left(z^2 \partial_z^2 + z\partial_z - \nu^2 - z^2 \right) \phi(z) = 0, \quad \text{with} \quad \nu^2 = (p_1 - p_2)^2, \tag{5.36}$$

whose linearly independent solutions are given by modified Bessel functions $I_\nu(z)$ and $K_\nu(z)$ of the first kind. The total solution in the original t-coordinate is therefore a linear combination of

$$\psi_R^\lambda(t; p_1, p_2) = \begin{cases} t^{\frac{p}{2}} I_\nu \left(\frac{2\Lambda}{\lambda\sqrt{t}} \right), \\ t^{\frac{p}{2}} K_\nu \left(\frac{2\Lambda}{\lambda\sqrt{t}} \right), \end{cases} \tag{5.37}$$

where $p = p_1 + p_2$. These modified Bessel functions have different asymptotics at infinity and relate to each other by going around the punctured disc $\mathbb{C}_t^*$.

The second order differential operator P_R defines the $\mathcal{D}_\lambda$-module

$$\mathcal{M}_R = \frac{\mathcal{D}_\lambda}{\mathcal{D}_\lambda \cdot P_R}, \tag{5.38}$$

which we claim represents fermions on the quantum SU(2) Seiberg-Witten geometry. To check this statement, we have to find the fermionic state corresponding to $\mathcal{M}_R$. So we asymptotically expand of the modified Bessel functions around $t = 0$ in λ:

$$\begin{aligned} I_\nu \left(\frac{2\Lambda}{\lambda\sqrt{t}} \right) &\sim t^{1/4} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}} \right) \left\{ 1 - \frac{(\mu-1)}{8} \frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu-1)(\mu-9)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \ldots \right\} \\ K_\nu \left(\frac{2\Lambda}{\lambda\sqrt{t}} \right) &\sim t^{1/4} \exp\left(-\frac{2\Lambda}{\lambda\sqrt{t}} \right) \left\{ 1 + \frac{(\mu-1)}{8} \frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu-1)(\mu-9)}{2! \cdot 8^2} \frac{\lambda^2 t}{4\Lambda^2} + \ldots \right\}, \end{aligned}$$

with $\mu = 4\nu^2$.

Recall that equation (5.22) implies that any exponential function in the local coordinate $v^{-1} = \sqrt{t}$ near the puncture acts trivially on the vacuum state. Equivalently, this is true for any asymptotic series in $\sqrt{t}$ that assumes the value 1 at $\sqrt{t} = 0$. In other words, we can forget about the complete expansion in $\sqrt{t}$! Only the WKB pieces

$$t^{1/4} \exp\left(\pm \frac{2\Lambda}{\lambda\sqrt{t}} \right) \tag{5.39}$$

are relevant in writing down the fermionic state. This is exactly opposite to the matrix model examples, where the WKB-piece can be neglected and the perturbative series in λ defines the fermionic state.

The derivatives of the above solutions have one term proportional to $\psi(s)$ (which we may forget about), and a term proportional to the derivative of the Bessel functions. The latter may be expanded as

$$\partial_s I_\nu(t) \sim t^{-1/4} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right)\left\{1 - \frac{(\mu+3)}{8}\frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu-1)(\mu+15)}{2!\cdot 8^2}\frac{\lambda^2 t}{4\Lambda^2} + \ldots\right\}$$
$$\partial_s K_\nu(t) \sim t^{-1/4} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right)\left\{1 + \frac{(\mu+3)}{8}\frac{\lambda\sqrt{t}}{2\Lambda} + \frac{(\mu-1)(\mu+15)}{2!\cdot 8^2}\frac{\lambda^2 t}{4\Lambda^2} + \ldots\right\}$$

around $\sqrt{t} = 0$. Again with the same reasoning only the WKB piece is necessary to write down the quantum state. Taking into account the extra factor $t^{\frac{p}{2}}$ in (5.37) the subspace $\mathcal{W}_R^+$ is thus generated by the $\mathcal{O}(t)$-module

$$t^{\frac{p}{2}} \begin{pmatrix} t^{\frac{1}{4}} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \\ t^{-\frac{1}{4}} \exp\left(\frac{2\Lambda}{\lambda\sqrt{t}}\right) \end{pmatrix} \mathcal{O}(t), \tag{5.40}$$

and blends (via the lexicographical ordening) into the fermionic state

$$|\mathcal{W}_R^+\rangle = v^{-p}\, e^{\frac{\Lambda}{\lambda}v}\left(v^{\frac{1}{2}} \wedge v^{-\frac{1}{2}} \wedge v^{-\frac{3}{2}} \wedge v^{-\frac{5}{2}} \wedge \ldots\right) \tag{5.41}$$

on the cover. Here we used a cover coordinate v^{-1} obeying $v^{-2} = t$, and rescaled the topological string coupling as $\tilde{\lambda} = \lambda/2$. $\mathcal{W}_R^+$ is thus simply generated by a single function

$$\psi^\lambda(v) = v^{-p} e^{\frac{\Lambda}{\lambda}v} \tag{5.42}$$

Hence the fermions blend into the Bogoliubov state

$$|\mathcal{W}_R^+\rangle = e^{\frac{\Lambda}{\lambda}\alpha_{-1}}|p\rangle, \tag{5.43}$$

when p is an integer.

Note that the only modulus that appears in this expression is p. This represents the diagonal U(1), denoting the total fermion flux through the geometry. The moduli p_1 and p_2 measure the fermion flux through an internal cycle and are not visible in the result, because the final state sums over all internal momenta. In general any SU(2) Seiberg-Witten geometry with the same quantized p yields the same fermionic state.

The fermionic (or dual) partition function is found by contracting the left and the right states, similarly as in the U(1) example above. The left state is just the complex conjugate of the right one, so we find

$$Z_{NO}^D(p;\lambda,\Lambda) = \langle p|e^{\frac{\Lambda}{\lambda}\alpha_1}e^{\frac{\Lambda}{\lambda}\alpha_{-1}}|p\rangle \sim \langle p|e^{\frac{1}{\lambda}\alpha_1}\Lambda^{2L_0}e^{\frac{1}{\lambda}\alpha_{-1}}|p\rangle. \tag{5.44}$$

The result is very similar to the U(1) example, up to the shift $\lambda \mapsto \lambda/2$. But notice that this fermionic state is written in terms of a single blended fermion. Decomposing this fermion into two twisted fermions makes it natural to insert an extra operator in the middle of the correlator, that measures the momenta of the two fermions through the A-cycles of the SW geometry. Weighting these momenta with a potential ξ_i, for $i = 1, 2$, yields

$$Z^D_{NO}(\xi_i, p; \lambda, \Lambda) \sim \langle p| e^{\frac{1}{\lambda}\alpha_1} e^{H_{\xi_i}} \Lambda^{2L_0} e^{\frac{1}{\lambda}\alpha_{-1}} |p\rangle, \tag{5.45}$$

where $H_{\xi_i} = \frac{1}{\lambda}\sum_r \xi_{(r+1/2) \bmod 2} \psi_r \psi^\dagger_{-r} = \frac{1}{\lambda}(p_1\xi_1 + p_2\xi_2)$. This is the answer conjectured by Nekrasov and Okounkov in [55].

5.2.3 U(N) theory

It is not difficult to extend this discussion to the U(N) theory (5.14), whose corresponding right half geometry we write as

$$\Sigma_N : \ \Lambda^N t^{-1} = \prod_{i=1}^{N} (v - u_i). \tag{5.46}$$

Canonically quantizing this geometry and changing the coordinates $z = \left(\frac{\Lambda}{\lambda}\right)^N t^{-1}$, brings us to the degree N differential equation

$$P_N \psi(z) = \left(\prod_{i=1}^{N} (z\partial_z - p_i) - z \right) \psi(z) = 0. \tag{5.47}$$

It turns out that a solution to the above equation is given by a particular Meijer G-function, denoted $G^{m,n}_{p,q}(z)$. The Meijer G-function is a complicated special function which was introduced in order to unify a number of standard special function [58–60], and is defined in terms of a complex integral

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \,\Big|\, z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - t) \prod_{j=1}^n \Gamma(1 - a_j + t)\, z^t}{\prod_{j=m+1}^q \Gamma(1 - b_j + t) \prod_{j=n+1}^p \Gamma(a_j - t)}\, dt, \tag{5.48}$$

where L is a contour which goes from $-i\infty$ to $+i\infty$ and separates the poles of $\Gamma(b_j - t)$, for $j = 1, \ldots, m$, from those of $\Gamma(1 - a_i + t)$, for $i = 1, \ldots, n$.

It can be shown that the Meijer G-function solves the differential equation

$$\left(\prod_{i=1}^{q} (z\partial_z - b_i) + (-1)^{p-m-n+1} z \prod_{j=1}^{p} (z\partial_z - a_j + 1) \right) G(z) = 0. \tag{5.49}$$

So, indeed the Seiberg-Witten differential equation (5.47) is a special case of Meijer differential equation (5.49) with $p = n = 0$ and $q = N$. Therefore the differential equation (5.47) is solved by

$$\psi(z) = G^{0,0}_{0,N}\left(\begin{matrix} \emptyset \\ p_1, p_2, \ldots, p_N \end{matrix} \,\Big|\, z \right). \tag{5.50}$$

Similarly as before we claim that the $\mathcal{D}$-module corresponding to U(N) Seiberg-Witten curve is generated by P_N. A subspace $\mathcal{W}$ corresponding to this $\mathcal{D}$-module is thus generated by a solution $\psi(t)$ and its derivatives in $t\partial_t$.

For $p < q$ the Meyer differential equation (5.49) has a regular singularity at $z = 0$ and an irregular one for $z = \infty$. To extract the I-brane fermionic state, we are interested in the behaviour around the irregular singularity, where $t \to 0$. It turns out that one of the independent solutions of the Seiberg-Witten differential equation (5.47) has the asymptotic expansion [58–60]

$$\psi(v) \sim e^{\frac{\Lambda}{\lambda/N} v} \, v^{\frac{1-N}{2}} \, v^p \sum_{j=0}^{\infty} k_j v^{-j}, \tag{5.51}$$

around this singularity, which is conveniently written in the cover coordinate $(-v)^N = t^{-1} = \left(\frac{\lambda}{\Lambda}\right)^N z$. The other solutions are found by multiplying the coordinate v by N-th roots of unity, and thus behave distinctly at infinity. As before, $p = \sum_{i=1}^N p_i$.

To find the fermionic state corresponding to the U(N) Seiberg-Witten curve, we act with $\psi(v)$ on the Dirac vacuum. The positive power of v in the exponent of $\psi(v)$ corresponds in the operator language to α_{-1}, whereas v^p lifts the Fermi level. The remaining series just contains negative powers of v which translate to a trivial action on the vacuum in the operator formalism. Therefore, the above asymptotic solution and its derivatives (in $t\partial_t$) blend into the state

$$|\mathcal{W}_R\rangle = e^{\frac{\Lambda}{\tilde{\lambda}}\alpha_{-1}} |p\rangle, \tag{5.52}$$

with rescaled topological string coupling $\tilde{\lambda} = \lambda/N$. Like for the U(2) Seiberg-Witten geometry the dependence on the individual moduli p_i has dropped out.

Similarly as in U(1) and U(2), in the present case we also find the U(N) Nekrasov-Okounkov dual partition function

$$Z^D_{NO}(\xi_i; \lambda, \Lambda) = \langle p | e^{\frac{1}{\lambda}\alpha_1} e^{H_{\xi_i}} \Lambda^{2L_0} e^{\frac{1}{\lambda}\alpha_{-1}} | p \rangle. \tag{5.53}$$

This fermionic correlator is indeed the one postulated in [55]. For $N = 1$ or $N = 2$ the Meijer G-function specializes respectively to the exponent and Bessel functions, which reproduces the results derived in previous subsections.

Although the normalizable moduli p_i disappear in the final I-brane partition function, they reappear when the state is unblended in terms of N single fermions

$$e^{\frac{1}{\lambda}\alpha_{-1}} |p\rangle = \sum_R \frac{\mu_R}{\tilde{\lambda}^{|R|}} |p, R\rangle = \sum_{\sum p_i = p} \sum_{R_{(i)}} \sqrt{Z^{pert}(p)} \, \frac{\mu_{\vec{R}}(p, \tilde{\lambda})}{\tilde{\lambda}^{|R|}} \bigotimes_{l=1}^{N} |p_i, R_{(i)}\rangle, \tag{5.54}$$

as may be seen from (5.10) and (5.13). The charges p_i have an interpretation as the fermion fluxes through the N tubes of the Seiberg-Witten geometry we started with.

Actually, we find the same fermionic state when starting with any other Seiberg-Witten geometry whose fermion flux at infinity is p. Hence one microstate in the total sum (5.54) can be interpreted as a fermion flux through an infinite set of geometries. This gives the state (5.54) as well as the partition function (5.8) the interpretation of a sum over geometries.

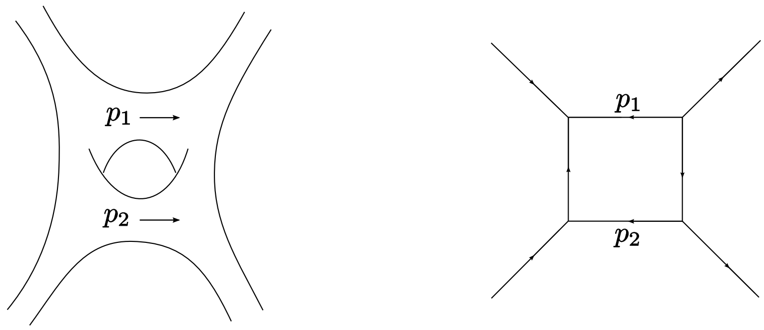

Figure 6. On the left we see the five-dimensional U(2) Seiberg-Witten surface with fermion fluxes through its A-cycles, and on the right a corresponding toric diagram. The fermion flux deforms the Kähler lengths of the toric diagram as in equation (5.55).

5.3 Relation to topological string theory

Nekrasov and Okounkov also derive a partition function for the 5-dimensional U(N) Seiberg-Witten theory compactified on the circle of circumference β [54, 55, 57] . It is given by a K-theoretic generalization of the 4-dimensional formula in equation (5.4).

This 5-dimensional theory is closely related to the topological string theory by geometric engineering [61] on a toric Calabi-Yau background [62, 63]. Namely, the partition function of the topological string theory on an A_N-singularity fibered over $\mathbb{P}^1$ (whose toric diagram consists of $N-1$ meshes as in figure 6) is equal to the partition function of the 5-dimensional gauge theory given above, when the Kähler sizes of the internal legs are

$$Q_{F_i} = e^{\beta(a_{i+1}-a_i)}, \qquad Q_B = \left(\frac{\beta\Lambda}{2}\right)^{2N}, \tag{5.55}$$

where F_i labels the vertical legs and B the horizontal ones. In the so-called gauge theory limit, when $\beta \to 0$, the topological string partition function reduces to the 4-dimensional Seiberg-Witten partition function. The corresponding B-model mirror geometry is of the form

$$X_{SW}: \quad xy - H(t,v) = 0, \tag{5.56}$$

where $H(t,v) = 0$ represents a Riemann surface of genus $N-1$. In the gauge theory limit this surface becomes the Seiberg-Witten curve Σ_{SW}, parametrized as in the equation (5.3).

In topological string theory it is natural as well to write down a dual partition function [4]. In a local B-model this allows the possibility of arbitrary fermion fluxes through the handles of the Riemann surface. In this setting it has been argued before that turning on a fermion flux is equivalent to deforming the geometry. More precisely, fermion flux parametrized by $\mathcal{P} = p_i \mathcal{B}_i$ changes the integral of the holomorphic 3-form over any linking 3-cycle $\mathcal{A}_i$, and thereby shifts the complex structure moduli $S_i = \int_{\mathcal{A}_i} \Omega$ as

$$S_i \mapsto S_i + \lambda p_i \tag{5.57}$$

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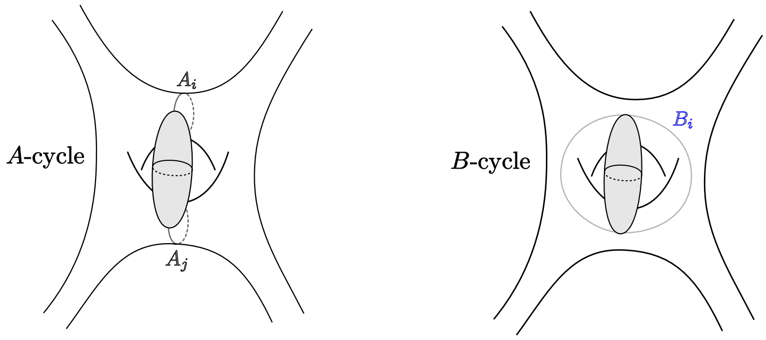

Figure 7. Three-cycles in the Seiberg-Witten U(2)-geometry.

In the A-model fermion flux translates into wrapping D4 branes around 4-cycles, and thereby deforms the Kähler moduli. The I-brane partition function thus equals the dual topological string partition function.

Because the Seiberg-Witten surface is embedded in $\mathbb{C}\times\mathbb{C}^*$, $\mathcal{A}$ and $\mathcal{B}$-cycles in the toric threefold will have topologies $S^1\times S^2$ and S^3, respectively (they are drawn in figure 7). In particular, a basis of $\mathcal{A}_i$-cycles can be chosen to reduce to the surface as the combination $A_i - A_{i+1}$. Now notice that the 3-cycle $\mathcal{A}_i$ with topology $S^1\times S^2$ is mirror to the vertical 2-cycle F_i that connects the i-th and the $i+1$-th horizontal leg. So turning on a fermion flux p_i through the i-th leg of the Seiberg-Witten geometry changes the complex structure parameter S_i by an amount proportional to $a_i - a_{i+1}$. This explains the Kähler size Q_{F_i} in (5.55) in terms of fermionic fluxes through the Seiberg-Witten curve, and in reverse why (5.54) may be interpreted as a sum over Seiberg-Witten geometries, or equivalently toric diagrams. So we conclude that the fermionic interpretation in 4d of Nekrasov and Okounkov is dual in 6d to the fermionic interpretation of the topological string, and has a deeper interpretation in terms of $\mathcal{D}$-modules.

5.3.1 Five-dimensional U(1) theory

Quantizing a five-dimensional Seiberg-Witten geometry yields a difference (instead of differential) equation. Working out $\mathcal{D}$-modules for these geometries we leave for future work. Let us treat one example in detail though. The five-dimensional right U(1) Seiberg-Witten half-geometry

$$\Sigma_R^{5d}: \quad \beta\Lambda e^{-\beta\lambda}t^{-1} + e^{-\beta v} - 1 = 0 \tag{5.58}$$

may be drawn as a pair of pants. In the field theory limit $\beta\to 0$ it reduces to the familiar equation $\Lambda t^{-1} = v$ for the right-half Seiberg-Witten geometry (with $u=0$).

In the B-model the most general state assigned to a local pair of pants geometry is given by a Bogoliubov state [4]

$$|\mathcal{W}\rangle = \exp\left[\sum_{i,j}\sum_{m,n=0}^{\infty} a^{ij}_{mn}\psi^i_{-m-1/2}\psi^{*j}_{-n-1/2}\right]|0\rangle, \tag{5.59}$$

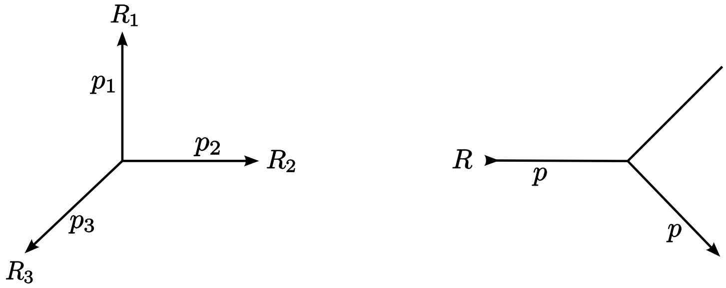

Figure 8. The B-model vertex (on the left) may be expanded as a sum over fermionic states $|p_1, R_1\rangle \otimes |p_2, R_2\rangle \otimes |p_3, R_3\rangle$, with $p_1 + p_2 + p_3 = 0$, corresponding to a conserved fermion flux through the pair of pants. The five-dimensional right-half Seiberg-Witten geometry (on the right) with charge p only has one partition $R \neq 0$.

where the index $i = 1, 2, 3$ describes the fermion field on the three asymptotic regions of the pair of pants, and the coefficients are determined by a comparison with the A-model topological vertex. This exponent can be expanded as a sum over states (see figure 8)

$$|p_1, R_1\rangle \otimes |p_2, R_2\rangle \otimes |p_3, R_3\rangle, \tag{5.60}$$

where the fermion flux is conserved: $p_1 + p_2 + p_3 = 0$. To describe the 5d Seiberg-Witten U(1) geometry we won't need this state in full generality.

The B-field quantizes this geometry into the difference equation

$$P(t)\Psi(t) = \left(\beta\Lambda e^{-\beta\lambda} t^{-1} + e^{\beta\lambda t\partial_t} - 1\right)\Psi(t) = 0. \tag{5.61}$$

Its fundamental solution is the quantum dilogarithm

$$\Psi(t) = \exp \sum_{n>0} \frac{(\beta\Lambda)^n t^{-n}}{n(1 - e^{\beta\lambda n})}. \tag{5.62}$$

As an intermezzo, notice that quantizing the equation

$$\beta v = -\log\left(1 - \beta\Lambda e^{-\beta\lambda} t^{-1}\right), \tag{5.63}$$

which is just a rewriting of equation (5.58) for Σ_R^{5d}, we find a differential equation which may be interpreted as the WKB approximation of difference equation (5.61). A fundamental solution of the differential equation is given by the genus 0 disc amplitude

$$\Psi_0(u) = \exp \sum_{n>0} \frac{(\beta\Lambda)^n t^{-n}}{\lambda n^2 e^{\beta\lambda n}}. \tag{5.64}$$

Acting with the five-dimensional dilogarithm on the Dirac vacuum state yields the fermionic state

$$|\mathcal{W}\rangle_{\mathrm{U}(1)}^{5d} = \exp \sum_{n>0} \frac{(\beta\Lambda)^n \alpha_{-n}}{n(1 - e^{\beta\lambda n})} |0\rangle. \tag{5.65}$$

This describes a subset of $|\mathcal{W}\rangle$ where only the quantum number R_1 is non-trivial. Summing over all external states of the form

$$|-p, R\rangle \otimes |p, \bullet\rangle \otimes |0, \bullet\rangle, \tag{5.66}$$

incorporates a fermion flux p through the pair of pants. In the field theory limit $\beta \to 0$ the resulting state reduces to the familiar four-dimensional state

$$\exp(\alpha_{-1}/\lambda)|p\rangle \otimes |p, \bullet\rangle \otimes |0, \bullet\rangle.$$

The partition function is found as the contraction of the left and right 5d half-geometries. (Or equivalently in the topological B-model by inserting a propagator [4].) This yields the fermionic correlator

$$\langle 0|\tilde{\Gamma}_+ \tilde{\Gamma}_-|0\rangle = \langle 0|\Gamma_+ (\beta\Lambda)^{2L_0} \Gamma_-|0\rangle, \tag{5.67}$$

with

$$\tilde{\Gamma}_\pm = \exp \sum_{\pm n > 0} \frac{(\beta\Lambda)^{|n|}\alpha_n}{|n|(1-e^{\beta\lambda n})} \quad \text{and} \quad \Gamma_\pm = \exp \sum_{\pm n > 0} \frac{\alpha_n}{|n|(1-e^{\beta\lambda n})}. \tag{5.68}$$

Indeed, the result equals the five-dimensional U(1) partition function

$$Z_{5d}^{\mathrm{U}(1)}(\lambda, \Lambda, \beta) = \exp \sum_{n=1}^{\infty} \frac{(\beta\Lambda)^{2n}}{4n \sinh^2(\beta\lambda n/2)}, \tag{5.69}$$

that was found by Nekrasov and Okounkov in [55].

6 Discussion

In this paper we argued that the fundamental objects underlying various systems in theoretical physics are chiral fermions living on quantum curves. In our formulation the quantum curve is defined, similarly to an affine classical curve, in terms of an equation of the form $P(z, w) = 0$. Its crucial feature, however, is the non-commutative character of the coordinates z, w. These quantum (or non-commutative) curves generalize the classical curves that come up in the standard formulation of a given topic. Examples of such classical curves can be found in the theory of random matrices, $c = 1$ string theory, Seiberg-Witten theory, and more generally in topological string theory. Semi-classically their (genus one) free energy is computed as a fermionic determinant on the classical curve. In our approach chiral fermions on the quantum curve generate the all-genus expansion of the free energy with respect to the non-commutativity parameter λ.

Fermions on a non-commutative curve can be realized physically within string theory as massless states of open strings on an intersecting brane configuration in the presence of a B-field. This idea was already put forward in [5]. In this paper we have exploited this I-brane system in a few important examples.

First of all we showed, reinterpreting the results in [38], that I-branes and $\mathcal{D}$-modules provide an insightful formulation of matrix models. This quite general statement is also appealing when certain matrix model limits are considered, such as a double scaling limits. In this case one recovers an I-brane formulation of minimal string theory and topological gravity. Secondly, we discussed how to reformulate $c = 1$ string theory in the framework of $\mathcal{D}$-modules.

Finally, we discussed supersymmetric gauge theories. Using $\mathcal{D}$-module formalism we derived fermionic expressions for the partition function of the $\mathcal{N} = 2$ gauge theory, reproducing the dual all-genus partition function introduced in [55]. We considered mainly 4-dimensional Seiberg-Witten geometries with unitary gauge groups, and explained only the simplest U(1) example in the 5-dimensional theory. It would be insightful to extend these results to other gauge groups and include matter content. It is clear that this should be possible, as these aspects of the 5-dimensional Seiberg-Witten theory are captured by topological string theory on toric manifolds. The latter system can be solved in fermionic B-model formulation of the topological vertex [4] which is equivalent to the I-brane fermions [5]. Nonetheless, finding the quantum I-brane curve representing such configurations appears to be a nontrivial task.

In all these examples we were able write down a $\mathcal{D}$-module that, through the prescription in section 2, yields the all-genus partition function. Especially the matrix model examples made it clear that this $\mathcal{D}$-module can be quite non-trivial in general. Only for the simplest curves, such as those appearing in double scaled matrix models, the $\mathcal{D}$-module can be found by canonically quantizing the classical spectral curve.

In the process of unraveling the $\mathcal{D}$-module structure in both sets of examples, we noticed some crucial differences. While the WKB piece of the $\mathcal{D}$-module generator can be ignored in finding the all-genus matrix model partition function, we discovered that it plays an eminent role for the Seiberg-Witten geometries. Another distinction is the difference in (non-)normalizable modes. While the potential W parametrizes non-normalizable modes that appear in the $\mathcal{D}$-module as parameters, in contrast, the normalizable modes in the Seiberg-Witten geometries are eaten by the $\mathcal{D}$-module, and only visible as a sum over internal fermion fluxes in the geometry. On the other hand, varying the $\mathcal{D}$-module with respect to the non-normalizable modes yields differential equations which relate to isomonodromy and the Stokes phenomenon.

Even with this rather broad set of examples, a few major questions remain. First of all, we cannot give a recipe in general how to find the quantum curve underlying a certain problem. Secondly, it is not obvious that our prescription is independent of the chosen parametrization of the classical curve. As we noted in the example of the classical curve $zw = 1$, different parametrizations can lead to different quantum curves that nonetheless yield the same partition function. This should hold in general cases as well, as topological string theory associates a unique all-genus partition function to a given curve. Thirdly, we haven't exploited some of the advantages of using $\mathcal{D}$-modules instead of differential equations. One of the main advantages is some independence on the way the differential equation is written down. It would be very interesting to try to match this freedom with the choice of parametrization for the classical curve. Finally, we have only discussed examples with one or two local patches. It would be highly insightful to study more general examples.

While in this paper our focus has been to associate a λ-perturbative quantum state to

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a spectral curve, we noticed that $\mathcal{D}$-modules in fact contain non-perturbative information. These bits get lost when we turn the $\mathcal{D}$-module in a fermionic state by making an asymptotic expansion of the $\mathcal{D}$-module generators in λ. This is in line with the discussion in [27], where it is argued that non-perturbative effects drastically modify the non-trivial target space curve into a complex plane. Non-perturbative effects in matrix models, as well as in the topological string theory, were also recently discussed in [64, 65]. Especially interesting in this respect is [66], where a non-perturbative partition function is proposed that is very similar to the I-brane partition function (2.12).

In the step where we turn a $\mathcal{D}$-module in a quantum state, a choice of boundary conditions has to be made. This implies that final states are troubled by the Stokes effect: solutions that decay faster can be added at no cost and the state changes when one crosses certain lines in the moduli space. This suggests that the $\mathcal{D}$-modules we studied in this paper may help in the understanding of wall-crossing phenomena in the corresponding $\mathcal{N}=2$ theories [67, 68].

More mathematically, our formalism is deeply connected with quantum integrable systems and the geometric Langlands program [8, 21, 22, 69–72], while approaching these topics from a string theoretic perspective. Especially interesting in this respect is our quantitative approach, that allows us to associate quantum invariants to spectral curves. In the future we hope to make this link even more concrete.

Specifically, it would be enlightening to have a better description of the non-commutative fermionic CFT on a given quantum curve. It is interesting to find out whether this relates to the WZW models based on opers in the geometric Langlands program: as so-called Hecke eigensheafs these generate examples of the Langlands correspondence. And, to discover the relation with the interacting bosonic CFT's that give another perspective on these intersecting brane configurations [73–75] as well as [76]. In particular, both models determine a set of recursion relations. It would be helpful to compare them.

A clear physical realization of quantum curves and the associated well-defined mathematical formulation in terms of $\mathcal{D}$-modules are great advantages of our approach. In consequence it can be applied to numerous situations mentioned above and yields definite quantitative results. Nonetheless, the idea of quantum curves is not new and earlier attempts of their formulation appeared before in physics and mathematics. It is worthwhile to recall how those attempts relate to our formalism.

The notion of quantum or non-commutative geometry has also been introduced by A. Connes [77]. His approach relies on replacing the algebra of functions on a manifold by a non-commutative C^*-algebra. In this context a program of developing a theory of non-commutative Riemann surfaces, from the point of view of geometric quantization [78], was advanced in [79]. Independently of this program, also some particular examples of low genus non-commutative Riemann surfaces have been analyzed in literature. In genus zero they include the so-called Podleś sphere [80] and more generally fuzzy spheres [81], which also found vast application in string theory. In genus one, one can consider a non-commutative torus which arises naturally in a certain realization of M-theory known as Matrix theory [82, 83]. Non-trivial B-field is an essential ingredient in a realization of these systems. It would be interesting to see if they could be related to I-brane configurations.

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A Infinite dimensional Grassmannian

In this section we introduce an infinite dimensional Grassmannian and its description in terms of the second quantized fermion field (we learned this material e.g. from [3, 13–16]).

A.1 Grassmannian and second quantized fermions

The space $\mathcal{H} = \mathbb{C}((z^{-1}))$ of all formal Laurent series in z^{-1} can be given an interpretation of a Hilbert space. Basis vectors z^n, for $n \in \mathbb{Z}$, correspond to one particle states of energy n associated to the Hamiltonian $z\partial_z$. This Hilbert space has a decomposition

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \tag{A.1}$$

such that the first factor $\mathcal{H}_+ = \mathbb{C}[z]$ is a subspace generated by z^0, z^1, z^2, ..., while $\mathcal{H}_-$ is generated by negative powers $z^{-1}, z^{-2}, \ldots$. Consider now a subspace $\mathcal{W}$ of $\mathcal{H}$ with a basis $\{w_k(z)\}_{k\in\mathbb{N}}$. We say it is comparable to $\mathcal{H}_+$, if in the projection onto positive and negative modes

$$w_k = \sum_{j\geq 0} (w_+)_{ij} z^j + \sum_{j>0} (w_-)_{ij} z^{-j} \tag{A.2}$$

the matrix w_+ is invertible. The Grassmannian Gr_0 is the set of all subspaces $\mathcal{W} \subset \mathbb{C}((z))$ which are comparable to $\mathcal{H}_+$.

In what follows we take much advantage of the correspondence between Gr_0 and the charge zero sector of the second quantized fermion Fock space $\mathcal{F}_0$. In this correspondence the subspace $\mathcal{H}_+$ is quantized as the Dirac vacuum

$$|0\rangle = z^0 \wedge z^1 \wedge z^2 \wedge \ldots, \tag{A.3}$$

with all positive energy states filled. The fermionic state associated to the subspace $\mathcal{W}$ with basis $w_0(z)$, $w_1(z)$, $w_2(z)$, ... is represented by the semi-infinite wedge[5]

$$|\mathcal{W}\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \ldots \tag{A.4}$$

[5]Actually, we have to tensor with $z^{\frac{1}{2}}$ to make the state fermionic.

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which is an element of the fiber of a determinant line bundle over the element $\mathcal{W} \in Gr$ (and therefore determined up a complex scalar c).

To make contact with the usual formulation of the second quantized fermion Fock space, we can identify the differentiation and wedging operators with the fermionic modes

$$\psi_{n+\frac{1}{2}} = \frac{\partial}{\partial z^{-n}} \qquad \psi^*_{n+\frac{1}{2}} = z^n \wedge . \tag{A.5}$$

These half-integer modes are annihilation and creation operators which arise from a decomposition of the fermion field $\psi(z)$ and its conjugate $\psi^*(z)$

$$\psi(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r z^{-r-\frac{1}{2}} \qquad \psi^*(z) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi^*_r z^{-r-\frac{1}{2}}, \tag{A.6}$$

and they obey the anti-commutation relations $\{\psi_r, \psi^*_{-s}\} = \delta_{r,s}$.

For subspaces $\mathcal{W} \in Gr_0$ the determinant of the projection onto $\mathcal{H}_+$ is well defined and can be expressed as

$$\det w_+ = \langle 0|\mathcal{W}\rangle. \tag{A.7}$$

More generally, one can consider the Fock space $\mathcal{F}$ which splits into subspaces of charge p

$$\mathcal{F} = \bigoplus_{p\in\mathbb{Z}} \mathcal{F}_p. \tag{A.8}$$

Each subspace $\mathcal{F}_p$ is built by acting with creation and annihilation operators on a vacuum

$$|p\rangle = z^p \wedge z^{p+1} \wedge z^{p+2} \wedge \ldots, \tag{A.9}$$

with the property

$$\begin{aligned} \psi_r|p\rangle &= 0 \qquad \text{for } r > p, \\ \psi^*_r|p\rangle &= 0 \qquad \text{for } r > -p. \end{aligned} \tag{A.10}$$

The Fermi level of the vacuum $|p\rangle$ is shifted by p units with respect to the Dirac vacuum $|0\rangle$. This fermion charge is measured by the U(1) current

$$J(z) =: \psi(z)\psi^*(z) := \sum_n \alpha_n z^{-n-1}, \tag{A.11}$$

whose components $\alpha_n = \sum_k : \psi_r \psi^*_{n-r}$ satisfy the bosonic commutation relations

$$[\alpha_m, \alpha_{-n}] = m\delta_{m,n}. \tag{A.12}$$

With each subspace $\mathcal{W} \subset \mathbb{C}((z))$ comparable to the one generated by $(z^k)_{k\geq p}$ one can associate a state $|\mathcal{W}\rangle \in \mathcal{F}$ of charge p. This charge is equal to the index of the projection operator $pr_+ : \mathcal{W} \to \mathcal{H}_+$.

A state in the Fock space $\mathcal{F}$ has also a simple representation in terms of the so-called Maya diagram (see figure 9). Black boxes in such a diagram represent excitations, whereas

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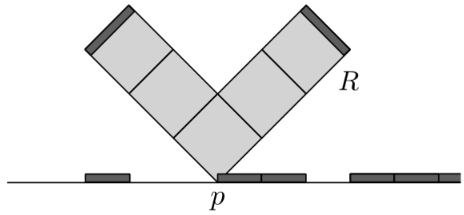

Figure 9. Elements of the Fock space $\mathcal{F}$ are in a bijective correspondence with Maya diagrams. The bottom line represent a Maya diagram corresponding to a fermionic state with charge p. As illustrated it is characterized by a two-dimensional partitions R located at position p. We therefore denote the state as $|p, R\rangle \in \mathcal{F}$.

white boxes are gaps in the energy spectrum of the fermion. The charge of a state is given by the number of excitations minus the number of gaps. Fermionic states or Maya diagrams of a fixed charge p can also be associated to two-dimensional partitions. In particular in $p = 0$ sector the state

$$|R\rangle = \prod_{i=1}^{d} \psi^*_{-a_i-\frac{1}{2}} \psi_{-b_i-\frac{1}{2}} |0\rangle \tag{A.13}$$

corresponds to the partition $R = (R_1, \ldots, R_l)$ such that

$$a_i = R_i - i, \qquad b_i = R_i^t - i. \tag{A.14}$$

In what follows a state corresponding to a partition R of charge p is denoted as $|p, R\rangle$.

A.2 Flow on the Grassmannian

There is an action on the Grassmannian defined by multiplying a basis vector $w_k(z)$ of $\mathcal{W}$ by a power series $f(z) = \sum f_n z^n$ that vanishes at $z = 0$.

$$f(z)|\mathcal{W}\rangle = \sum_k w_0 \wedge \ldots \omega_{k-1} \wedge f \cdot w_k \wedge w_{k+1} \ldots . \tag{A.15}$$

When we write $w_k(z)$ in terms of the basis $(z^l)_{l\in\mathbb{Z}}$ this action is encoded by the multiplication by an infinite matrix in gl_∞, whose $(i,j)^{\text{th}}$ entry is given by f_{i-j}. On the fermionic state $|\mathcal{W}\rangle$ a multiplication by z^n translates into a commutator with the bosonic mode α_n, since α_n increases the fermionic mode number by

$$[\alpha_n, \psi_r] = \psi_{r+n}. \tag{A.16}$$

Multiplication by a power series $f(z)$ therefore translates to the operator

$$f = \sum_n f_n [\alpha_n, \bullet] \ \in gl_\infty \tag{A.17}$$

on the Fock space.

Exponentiating the action of gl_∞ yields the group Gl_∞. An element $g(z) = \exp(f(z))$ of this group acts on $|\mathcal{W}\rangle$ by multiplying all its basis vectors

$$g(z)|\mathcal{W}\rangle = g \cdot w_0 \wedge \ldots \wedge g \cdot w_k \wedge \ldots . \tag{A.18}$$

From the fermionic point of view this action is given by conjugating each basis vector w_k with the element

$$g = \exp\left(\sum f_n \alpha_n\right) = \exp\left(\oint dz\ f(z)J(z)\right) \in Gl_\infty. \tag{A.19}$$

We call Γ the group of exponentials $g(z) : S^1 \to \mathbb{C}^*$. An important subgroup of Γ is the group Γ_+ of functions $g_0 : S^1 \to \mathbb{C}^*$ that extend over the disk $D_0 = \{z : |z| \leq 1\}$:

$$\Gamma_+ = \{g_0 : D_0 \to \mathbb{C}^* : g_0(0) = 1\}. \tag{A.20}$$

Another subgroup is the group Γ_- of functions $g_\infty : S^1 \to \mathbb{C}^*$ that extend over the disk $D_\infty = \{z \in \mathbb{C} \cup \{\infty\} : |z| \leq 1\}$:

$$\Gamma_- = \{g_\infty : D_\infty \to \mathbb{C}^* : g_\infty(\infty) = 1\}. \tag{A.21}$$

Any $g \in \Gamma$ can be written as an exponential $\exp(f)$. When $g \in \Gamma_+$ the function f vanishes at $z = 0$, and when $g \in \Gamma_-$ it vanishes at $z = \infty$.

Γ_+ and Γ_- have different properties when acting on Grassmannian. The action of Γ_- is free, since any $\mathcal{W} \in Gr$ has only a finite number of excitations. On the contrary, Γ_+ acts trivially on a vacuum state $|p\rangle$. Although the action of the groups Γ_+ and Γ_- on a subspace $\mathcal{W}$ is commutative, as it is just given by multiplication, as operators on the fermionic state $|\mathcal{W}\rangle$ it matters which element is applied first. This introduces normal ordering ambiguities.

An element

$$g(t, z) = \exp\left(\sum_{k\geq 1} t_k z^k\right) = \exp\left(f(t, z)\right) \in \Gamma_+, \tag{A.22}$$

defines a linear flow over the Grassmannian Gr. On the Fock space it acts as an evolution operator

$$U(t) = \exp\left(\oint \frac{dz}{2\pi i} f(t, z)J(z)\right). \tag{A.23}$$

The determinant $\det(\mathcal{W})_+$ is not equivariant with respect to the action of Γ_+. The difference is measured by the so-called tau-function

$$\tau_{\mathcal{W}}(g) = \frac{\det\left(g^{-1}w\right)_+}{g^{-1} \det\ w_+} = \frac{\langle 0|U(t)|\mathcal{W}\rangle}{g^{-1}\langle 0|\mathcal{W}\rangle}, \tag{A.24}$$

which yields a holomorphic function $\tau : \Gamma_+ \to \mathbb{C}$. This can be regarded as a wave function of $|\mathcal{W}\rangle$.

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A.3 Blending

So far we considered the Hilbert space $\mathcal{H} \equiv \mathcal{H}^{(1)}$ of functions with values in $\mathbb{C}$. More generally, one can consider a Hilbert space $\mathcal{H}^{(n)}$ of functions with values in $\mathbb{C}^n$. Let $(\epsilon_i)_{i=1,\ldots,n}$ denote a basis of $\mathbb{C}^n$. For each n there is an isomorphism between $\mathcal{H}^{(n)}$ and $\mathcal{H}$ given by the lexicographical identification of the basis

$$\epsilon_i z^k \mapsto z^{nk+i-1}. \tag{A.25}$$

This isomorphism is called blending.

In the fermionic language the Hilbert space $\mathcal{H}^{(n)}$ lifts to the Fock space of n fermions $\psi^{(i)}$, $i = 1, \ldots, n$, each one with the expansion (A.6) and such that

$$\{\psi_r^{(i)}, \psi_s^{*\,(j)}\} = \delta_{i,j}\delta_{r,-s}. \tag{A.26}$$

Now blending translates to the following redefinitions of these n fermions into a single fermion ψ

$$\psi_{n(r+\rho_i)} = \psi_r^{(i)}, \qquad\qquad \psi^*_{n(r-\rho_i)} = \psi_r^{*\,(i)}, \tag{A.27}$$

where

$$\rho_i = \frac{2i-n-1}{2n}. \tag{A.28}$$

Blending can also be expressed in terms of two-dimensional partitions introduced above. Consider n partitions $R_{(i)}$ of charges p_i, with $\sum_i p_i = p$, corresponding to states in n independent Hilbert spaces of fermions $\psi^{(i)}$. Associating with each such partition a state of a chiral fermion $|p_i, R_{(i)}\rangle$, we have a decomposition

$$|p, \mathbf{R}\rangle = \bigotimes_{i=1}^{n} |p_i, R_{(i)}\rangle, \tag{A.29}$$

and the blended partition $\mathbf{R}$ of charge p, corresponding to a state in the Hilbert space of the blended fermion Ψ, is defined as

$$\{n(p_i + R_{(i),m} - m) + i - 1 \mid m \in \mathbb{N}\} = \{p + \mathbf{R}_K - K \mid K \in \mathbb{N}\}. \tag{A.30}$$

B Some background on $\mathcal{D}$-modules

The theory of $\mathcal{D}$-modules was introduced and developed, among others, by I. Bernstein, M. Kashiwara, T. Kawai and M. Sato, to study linear partial differential equations from an algebraic perspective [9–12]. Currently this is a very active field, with connections and applications to many other branches of mathematics.

$\mathcal{D}$-modules are defined as modules for the algebra of differential operators $\mathcal{D}$. In general, in a local $\mathbb{C}^n$ patch with complex coordinates $(z_1, \ldots, z_n)$, the operators z_i and ∂_{z_i} represent the n^{th} Weyl algebra. The operators $P \in \mathcal{D}$ are of the form

$$P = \sum_{i_1,\ldots,i_n} a_{i_1,\ldots,i_n} \partial_{z_{i_1}} \cdots \partial_{z_{i_n}}. \tag{B.1}$$

With a set of operators $P_1, \ldots, P_m \in \mathcal{D}$ one can associate a system of differential equations

$$P_1\Psi = \ldots = P_m\Psi = 0, \tag{B.2}$$

where Ψ takes values in some function space $\mathcal{V}$. An algebraic description of solutions to such a system can be given in terms of a $\mathcal{D}$-module $\mathcal{M}$ determined by an ideal generated by $P_1, \ldots, P_m \in \mathcal{D}$

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot \langle P_1, \ldots, P_m \rangle}. \tag{B.3}$$

The advantage of considering such a $\mathcal{D}$-module is, firstly, that it captures the solutions to the above system of differential equations independently of the form in which this system is written. Secondly, it is also independent of the function space $\mathcal{V}$ — be it the space of square-integrable functions, the space of distributions, the space of holomorphic functions, etc.

Nonetheless, having chosen a particular space $\mathcal{V}$ one is interested in, the space of solutions is simply given by the algebra homomorphism

$$\mathrm{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{V}). \tag{B.4}$$

E.g. holomorphic solutions to the differential equation $P\Psi(z) = 0$ can be captured as a homomorphism of $\mathcal{D}$-modules

$$\mathcal{M} = \frac{\mathcal{D}}{\mathcal{D} \cdot P} \ \rightarrow \ \mathcal{O}_{\mathbb{C}}, \tag{B.5}$$

with $\mathcal{O}_{\mathbb{C}}$ the algebra of holomorphic functions on the complex plane $\mathbb{C}$. Indeed, define a map that sends the element

$$[1] \in \mathcal{M} \ \mapsto \ \Psi(z) \in \mathcal{O}_{\mathbb{C}}. \tag{B.6}$$

This is well-defined because every element $P' \in \mathcal{D}P$ is mapped to zero (remember that Ψ fulfills $P\Psi = 0$), and it is a bijection; conversely, any map $\mathcal{M}$ to $\mathcal{O}_{\mathbb{C}}$ is determined by a holomorphic solution to the differential equation $P\Psi = 0$.

An important notion is a dimension of a $\mathcal{D}$-module. The so-called Bernstein inequality asserts that a non-zero $\mathcal{D}$-module $\mathcal{M}$ over the n^{th} Weyl algebra has a dimension $2n \geq \dim\mathcal{M} \geq n$. In particular, $\mathcal{D}$ considered itself as a $\mathcal{D}$-module has a dimension $2n$. On the other hand, $\dim \mathbb{C}[x_1, \ldots, x_n] = n$. For a non-zero $P \in \mathcal{D}$, $\dim\mathcal{D}/\mathcal{D}P = 2n - 1$.

A special role in the theory of $\mathcal{D}$-modules is played by the so-called holonomic $\mathcal{D}$-modules, which by definition have a minimal dimension n. In particular they are cyclic, which means of the form $\{D\Psi : D \in \mathcal{D}\}$, i.e. they are determined by a single element $\Psi \in \mathcal{M}$ called a generator.

In the context of the I-brane in $\mathbb{C}^2$ we are just interested in the 1^{st} Weyl algebra (2.15) of dimension 2. In this case we immediately conclude that the module $\mathcal{D}/\mathcal{D}P$ has a dimension $n = 1$ for any non-zero P, and is thus holonomic and cyclic. It can be realized as

$$\mathcal{M} = \{D\Psi : D \in \mathcal{D}\}, \tag{B.7}$$

where the generator Ψ is a solution to the differential equation $P\Psi = 0$.

B.1 Flat connections

More generally, $\mathcal{D}$-modules are defined as differential sheaves on any variety X. The sections of the sheaf $\mathcal{D}_X$ over an open neighbourhood U are given by linear differential operators on U. Therefore, both the structure sheaf $\mathcal{O}_X$ (of holomorphic functions) as well as the tangent sheaf T_X (whose local sections are vector fields) may be embedded in $\mathcal{D}_X$

$$\mathcal{O}_X \hookrightarrow \mathcal{D}_X \hookleftarrow T_X. \tag{B.8}$$

In fact, $\mathcal{D}_X$ is generated by these inclusions.

A sheaf $\mathcal{M}$ on X is defined to be a left module for $\mathcal{D}_X$ when $v \cdot s \in \mathcal{M}$, for any $v \in \mathcal{D}_X$ and $s \in \mathcal{M}$. Furthermore, it has to fulfill

$$\begin{aligned} v \cdot (fs) &= v(f)s + f(v \cdot s) \\ [v, w] \cdot s &= v \cdot (w \cdot s) - w \cdot (v \cdot s) \end{aligned} \tag{B.9}$$

for any $v \in \mathcal{D}_X$, $f \in \mathcal{O}_X$ and $s \in \mathcal{M}$. Suppose that $\mathcal{M}$ is a left $\mathcal{D}_X$-module whose sections are the local sections of some vector bundle V (this encomprises all $\mathcal{D}_X$-modules that are finitely generated as $\mathcal{O}_X$-modules). Then the action of $\mathcal{D}_X$ defines a connection on V as

$$\nabla_v(s) = v \cdot s, \tag{B.10}$$

whose curvature is zero. So a $\mathcal{D}$-module structure on the sheaf of sections of a vector bundle V defines a flat connection on this vector bundle. And conversely, any module consisting of sections of a vector bundle V with flat connection ∇_A, has an interpretation as a $\mathcal{D}$-module defined through the action of the flat connection. Therefore, a $\mathcal{D}$-module is in general just a system of linear differential equations, changing from patch to patch on X. This is known as a local system. In the main part of this paper X is just $\mathbb{C}$ or $\mathbb{C}^*$.

C Relation to quantum integrable systems

In this article we focus on smooth curves that are given by an equation of the form

$$\Sigma: \quad H(z, w) = w^n + u_{n-1}(z) w^{n-1} + \ldots + u_0(z) = 0, \tag{C.1}$$

where $z \in \mathbb{C}$ (or $\mathbb{C}^*$) and $w \in \mathbb{C}$. These play a prominent role in integrable systems as spectral curves. It is a degree n cover over $\mathbb{C}$ (or $\mathbb{C}^*$)

$$\begin{array}{c} \Sigma \subset T^*\mathbb{C} \\ \downarrow \pi \\ \mathbb{C} \end{array} \tag{C.2}$$

with possible branch points (from now on we restrict to $z \in \mathbb{C}$ for simplicity in notation). The spectral curve is embedded in $\mathbb{C}^2$ and equipped with the (meromorphic) 1-form

$$\eta = \frac{1}{\lambda} w dz|_{\Sigma}. \tag{C.3}$$

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Our notion of a quantum curve agrees with a notion of quantum spectral curves in this context. Let us say a few words about this.

Fermions on Σ transform as holomorphic sections of a line bundle $L \otimes K_\Sigma^{1/2}$, provided by the D6-brane. The pair (L, η) on Σ pushes forward to a couple

$$\pi_* : \ (L, \eta) \mapsto (V = \pi_* L, \phi = \pi_* \eta) \tag{C.4}$$

on $\mathbb{C}$ under the projection map $\pi : \Sigma \to \mathbb{C}$. So V is a rank n vector bundle on $\mathbb{C}$, whereas ϕ is a holomorphic 1-form valued in $gl(n)$.[6] Such an object is called a Higgs field. It endows V with the structure of a Higgs bundle. Setting the characteristic polynomial

$$\det(\eta - \phi(z)) = 0 \tag{C.5}$$

returns the equation for the spectral curve. The push-forward map π_* sets up a bijection between spectral data and (stable) Higgs pairs

$$(\Sigma, L) \ \leftrightarrow \ (V, \phi). \tag{C.6}$$

The moduli space of stable Higgs pairs is an algebraically completely integrable system, known as the Hitchin integrable system

A $\mathcal{D}_\lambda$-module (as in [20]) corresponds to a λ-connection ∇_λ

$$\nabla_\lambda = \lambda \partial_z - A(z), \tag{C.7}$$

which is defined through the Leibnitz rule $\nabla_\lambda(fs) = f\nabla_\lambda(s) + \lambda s \otimes df$ for any function f and section s.

Semi-classically, such a λ-connection ∇_λ reduces to a 1-form $\nabla_0(z)$ with values in $gl(n)$

$$\nabla_\lambda \mapsto \nabla_0, \quad (\lambda \to 0). \tag{C.8}$$

We just encountered this object as a Higgs field ϕ. Moreover, we explained with (C.4) that a Higgs (V, ϕ) and spectral data (Σ, L) provide equivalent information. In particular, the spectral curve can be recovered by the determinant of the Higgs field. This implies that λ-connections quantize spectral data.[7]

It tells us exactly which requirements a $\mathcal{D}$-module quantizing the I-brane configuration has to satisfy. Fermions on a degree n spectral curve have to transform under a rank n λ-connection ∇_λ on $\mathbb{C}$, whose semi-classical $\lambda \to 0$ limit is given by the Higgs field

$$\nabla_0 = \pi_*(\eta). \tag{C.9}$$

A simple example of a λ-connection is given by

$$\nabla = \lambda \partial_z - A(z), \tag{C.10}$$

with $A(z) = \pi_*(\eta)$. Its determinant is a degree n differential equation that canonically quantizes the defining equation for Σ.

[6] In other words, $\phi \in H^0(\mathbb{C}, \mathrm{End}V \otimes K_\mathbb{C})$.

[7] These λ-connections are also known as λ-opers, and play an important role in the quantum integrable system of Beilinson and Drinfeld [21, 22].

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